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Self-stabilizing Leader Election in Polynomial Steps *

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Abstract

In this report, we propose a silent self-stabilizing leader election algorithm for bidirectional connected identified networks of arbitrary topology. This algorithm is written in the locally shared memory model. It assumes the distributed unfair daemon, the most general scheduling hypothesis of the model. Our algorithm requires no global knowledge on the network (such as an upper bound on the diameter or the number of processes, for example).

We show that its stabilization time is in $\Theta(n^3)$ steps in the worst case, where n is the number of processes. Its memory requirement is asymptotically optimal, *i.e.*, $\Theta(\log n)$ bits per processes. Its round complexity is of the same order of magnitude — *i.e.*, $\Theta(n)$ rounds — as the best existing algorithm [9] designed with similar settings (*i.e.*, it does not use global knowledge and is proven under the unfair daemon).

To the best of our knowledge, this is the first self-stabilizing leader election algorithm for arbitrary identified networks that is proved to achieve a stabilization time polynomial in steps. By contrast, we show that the previous best existing algorithm designed with similar settings [9] stabilizes in a non polynomial number of steps in the worst case.

Keywords: Distributed algorithms, fault-tolerance, self-stabilization, leader election, unfair daemon.

Résumé

Dans ce rapport, nous présentons un algorithme d'élection silencieux et auto-stabilisant pour un réseau bidirectionnel et connecté de topologie quelconque. Cet algorithme utilise le modèle à états. Il fonctionne sous un démon inéquitable qui est l'hypothèse d'ordonnancement la plus faible de ce modèle. Notre algorithme ne nécessite *a priori* aucune connaissance globale sur le réseau (comme par exemple une borne supérieure sur le diamètre ou le nombre de processus).

Nous montrons que cet algorithme stabilise en $\Theta(n^3)$ pas de calcul dans le pire des cas, où n est le nombre de processus. La mémoire nécessaire est asymptotiquement optimale, *i.e.*, $\Theta(\log n)$ bits par processus. Sa complexité en rondes est du même ordre de grandeur – *i.e.*, $\Theta(n)$ rondes – que celle du meilleur algorithme existant [9] avec les mêmes hypothèses (cet algorithme n'utilise aucune connaissance globale et est prouvé sous l'hypothse d'un démon inéquitable).

A notre connaissance, il s'agit du premier algorithme auto-stabilisant d'élection pour des réseaux identifiés quelconques prouvé comme ayant un temps de stabilisation polynomial en nombre de pas. Pour comparer, nous montrons que le temps de stabilisation du meilleur algorithme (avant ce travail) utilisant les mêmes hypothèses [9] est non polynomial en nombre de pas.

Mots-clés : Algorithmes répartis, tolérance aux pannes, auto-stabilisation, élection, démon inéquitable.

Contents

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Introduction

In distributed computing, the *leader election* problem consists in distinguishing one process, so-called the leader, among the others. We consider here identified networks. So, as it is usually done, we augment the problem by requiring all processes to eventually know the identifier of the leader. The leader election is fundamental as it is a basic component to solve many other important problems, *e.g.*, consensus, spanning tree constructions, implementing broadcasting and convergecasting methods, *etc*.

Self-stabilization [10, 11] is a versatile technique to withstand *any* transient fault in a distributed system: a self-stabilizing algorithm is able to recover, *i.e.*, reach a legitimate configuration, in finite time, regardless the *arbitrary* initial configuration of the system, and therefore also after the occurrence of transient faults. Thus, self-stabilization makes no hypotheses on the nature or extent of transient faults that could hit the system, and recovers from the effects of those faults in a unified manner. Such versatility comes at a price. After transient faults, there is a finite period of time, called the *stabilization phase*, before the system returns to a legitimate configuration. The *stabilization time* is then the worst case duration of the stabilization phase, *i.e.*, the maximum time to reach a legitimate configuration starting from an arbitrary one. Notice that efficiency of self-stabilizing algorithms is mainly evaluated according to their stabilization time and memory requirement.

We consider the deterministic [1] asynchronous silent self-stabilizing leader election problem in bidirectional, connected, and identified networks of arbitrary topology. We investigate solutions to this problem which are written in the locally shared memory model introduced by Dijkstra [10]. In this model, the distributed unfair daemon is known as the weakest scheduling assumption. Now, proving the self-stabilization of a given algorithm under such an assumption implies that the stabilization time is finite in terms of atomic steps. However, despite some solutions assuming all these settings (in particular the unfairness assumption) are available in the literature [7, 8, 9], none of them is proven to achieve a polynomial upper bound in steps on its stabilization time. Rather, the time complexities of all these solutions are analyzed in terms of rounds only.

Related Work.

In [12], Dolev *et al* showed that the silent self-stabilizing leader election requires $\Omega(\log n)$ bits per process, where n is the number of processes. Self-stabilizing leader election algorithms for arbitrary connected identified networks have been proposed in the message-passing model [1, 4, 5]. First, the algorithm of Afek and Bremler [1] stabilizes in $O(n)$ rounds using $\Theta(\log n)$ bits per process. But, it assumes that the link-capacity is bounded by a value B, known by all processes. Two solutions that stabilize in $O(\mathcal{D})$ rounds, where $\mathcal{D}$ is the diameter of the network, have been proposed in [4, 5]. However, both solutions assume that processes know some upper bound D on the diameter $\mathcal{D}$; and have a memory requirement in $\Theta(\log D \log n)$ bits.

Several solutions are also given in the shared memory model [13, 3, 7, 8, 9, 15]. The algorithm proposed by Dolev and Herman [13] is not silent, works under a *fair* daemon, and assume that all processes know a bound N on the number of processes. This solution stabilizes

1. We only consider here deterministic algorithms.

in $O(\mathcal{D})$ rounds using $\Theta(N \log N)$ bits per process. The algorithm of Arora and Gouda [3] works under a *weakly fair* daemon and assume the knowledge of some bound N on the number of processes. This solution stabilizes in $O(N)$ rounds using $\Theta(\log N)$ bits per process.

Datta *et al* [7] propose the first self-stabilizing leader election algorithm (for arbitrary connected identified networks) proven under the distributed unfair daemon. This algorithm stabilizes in $O(n)$ rounds. However, the space complexity of this algorithm is unbounded. (More precisely, the algorithm requires each process to maintain an unbounded integer in its local memory.)

Solutions in [8, 9, 15] have a memory requirement which is asymptotically optimal (*i.e.* in $\Theta(\log n)$). The algorithm proposed by Kravchik and Kutten [15] assumes a synchronous daemon and the stabilization time of this latter is in $O(\mathcal{D})$ rounds. The two solutions proposed by Datta *et al* in [8, 9] assume a distributed unfair daemon and have a stabilization time in $O(n)$ rounds. However, despite these two algorithms stabilize within a finite number of step (indeed, they are proved assuming an unfair daemon), no step complexity analysis is proposed. Finally, note that the algorithm proposed in [8] assumes that each process has a bit of memory which cannot be arbitrarily corrupted.

Contribution.

We propose a silent self-stabilizing leader election algorithm for arbitrary connected and identified networks. Our solution is written in the locally shared memory model assuming a distributed unfair daemon, the weakest scheduling assumption. Our algorithm assumes no knowledge of any global parameter (*e.g.*, an upper bound on $\mathcal{D}$ or n) of network. Like previous solutions of the literature [8, 9], it is asymptotically optimal in space (*i.e.*, it works using $\Theta(\log n)$ bits per process), and it stabilizes in $\Theta(n)$ rounds in the worst case. Yet, contrary to those solutions, we show that our algorithm has a stabilization time in $\Theta(n^3)$ steps in the worst case.

For fair comparison, we have also studied the step complexity of the algorithm given in [9], noted here $\mathcal{DLV}$. This latter is the closest to ours in terms of performance. We show that its stabilization time is not polynomial, *i.e.*, there is no constant α such that the stabilization time of $\mathcal{DLV}$ is in $O(n^\alpha)$ steps. More precisely, we show that fixing α to any constant greater than or equal to 4, for every $\beta \geq 2$, there exists a network of $n = 2^{\alpha-1} \times \beta$ processes in which there exists a possible execution that stabilizes in $\Omega(n^\alpha)$ steps.

Roadmap.

The next chapter is dedicated to computational model and basic definitions. In Chapter 3, we propose our self-stabilizing leader election algorithm. We prove its correctness in Chapter 4. In the same chapter, we also study its stabilization time in both steps and rounds. We show that the stabilization time of the self-stabilizing leader election algorithm given in [9] is not polynomial in steps in Chapter 5. We conclude in Chapter 6.

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Computational model

2.1 Distributed systems

We consider *distributed systems* made of n *processes*. Each process can communicate with a subset of other processes, called its *neighbors*. We denote by $\mathcal{N}_p$ the set of neighbors of process p. Communications are assumed to be bidirectional, *i.e.* $q \in \mathcal{N}_p$ if and only if $p \in \mathcal{N}_q$. Hence, the topology of the system can be represented as a simple undirected connected graph $G = (V, E)$, where V is the set of processes and E is a set of edges representing (direct) communication relations. We assume that each process has a unique ID, a natural integer. IDs are stored using a constant number of bits, b. As commonly done in the literature, we assume that $b = \Theta(\log n)$. Moreover, by an abuse of notation, we identify a process with its ID, whenever convenient. We will also denote by ℓ the process of minimum ID. (So, the minimum ID will be also denoted by ℓ.)

2.2 Locally shared memory model

We consider the *locally shared memory model*, in which the processes communicate using a finite number of locally shared registers, called *variables*. Each process can read its own variables and those of its neighbors, but can write only to its own variables. The *state* of a process is the vector of the values of all its variables. A configuration γ of the system is the vector of the states of all processes. We denote by $\gamma(p)$ the state of the process p in the configuration γ. We denote by $\mathcal{C}$ the set of all possible configurations.

A distributed *algorithm* consists of one *program* per process. The program of a process p is a finite set of actions of the following form:

$$\langle\text{label}\rangle :: \langle\text{guard}\rangle \to \langle\text{statement}\rangle$$

The *labels* are used to identify actions. The *guard* of an action in the program of process p is a Boolean expression involving the variables of p and its neighbors. If the guard of some action evaluates to true, then the action is said to be *enabled* at p. By extension, if at least one action is enabled at p, p is said to be enabled. We denote by $Enabled(\gamma)$ the set of processes enabled in configuration γ. The *statement* of an action is a sequence of assignments on the variables of p. An action can be executed only when it is enabled. In this case, the execution of the action consists in executing its statement.

The asynchronism of the system is materialized by an adversary, called the *daemon*. In a configuration γ, if $Enabled(\gamma) \neq \emptyset$, then the daemon selects a non empty subset S of $Enabled(\gamma)$ to perform an *atomic step*: $\forall p \in S$, p atomically executes one of its actions enabled in γ, leading the system to a new configuration γ'. We denote by $\mapsto$ the relation between configurations such that $\gamma \mapsto \gamma'$ if and only if γ' can be reached from γ in one atomic step. An *execution* is then a *maximal* sequence of configurations $\gamma_0, \gamma_1, \ldots$ such that $\gamma_{i-1} \mapsto \gamma_i, \forall i > 0$. The term "maximal" means that the execution is either infinite, or ends at a *terminal* configuration γ in which $Enabled(\gamma)$ is empty.

As we saw previously, each step from a configuration to another is driven by a daemon. In this paper, the daemon is supposed to be *distributed* and *unfair*. "Distributed" means that while the configuration is not terminal, the daemon should select at least one enabled process, maybe more. "Unfair" means that there is no fairness constraint, *i.e.*, the daemon might never permit an enabled process to execute, unless it is the only enabled process.

2.3 Rounds

To measure the time complexity of an algorithm, we also use the notion of *round*. This latter allows to highlight the execution time according to the speed of the slowest process. If a process p is enabled in a configuration γ_i but not enabled in the next configuration γ_{i+1} and does not execute any action between γ_i and γ_{i+1}, we said that p is *neutralized* during the step $\gamma_i \mapsto \gamma_{i+1}$. Neutralization of p is caused by the following situation: at least one neighbor of p changes its state between γ_i and γ_{i+1}, and this change makes the guards of all actions of p false. The first round of an execution e, noted e', is the minimal prefix of e in which every process that is enabled in the initial configuration either executes an action or becomes neutralized. Let e'' be the suffix of e starting from the last configuration of e'. The second round of e is the first round of e'', and so forth.

2.4 Self-Stabilization

Let $\mathcal{A}$ be a distributed algorithm. Let $\mathcal{E}$ be the set of all possible executions of $\mathcal{A}$. A *specification* SP is a predicate over $\mathcal{E}$.

$\mathcal{A}$ is *self-stabilizing* for SP if and only if there exists a non-empty subset of configurations $\mathcal{L} \subseteq \mathcal{C}$, called *legitimate* configurations, such that:

– *Closure:* $\forall e \in \mathcal{E}$, for each step $\gamma_i \mapsto \gamma_{i+1} \in e$, $\gamma_i \in \mathcal{L} \Rightarrow \gamma_{i+1} \in \mathcal{L}$.

– *Convergence:* $\forall e \in \mathcal{E}, \exists \gamma \in e$ such that $\gamma \in \mathcal{L}$.

– *Correction:* $\forall e \in \mathcal{E}$ such that e starts in a legitimate configuration $\gamma \in \mathcal{L}$, e satisfies SP.

Every configuration that is not legitimate is called *illegitimate*. The *stabilization time* is the maximum time (in steps or rounds) to reach a legitimate configuration starting from any configuration.

2.5 Self-Stabilizing Leader Election

We define $SP_{LE}(e)$ the specification of the leader election problem. Let $Leader : V \mapsto \mathbb{N}$ be a function defined on the state of any process $p \in V$ in the current configuration that returns the ID of the leader appointed by p. $SP_{LE}(e)$ is true if and only if:

1. For all configuration $\gamma \in e$, $\forall p, q \in V$, $Leader(p) = Leader(q)$ and $Leader(p)$ is the ID of some process in V.
2. For all step $\gamma_i \mapsto \gamma_{i+1} \in e$, $\forall p \in V$, $Leader(p)$ has the same value in γ_i and γ_{i+1}.

$\mathcal{A}$ is *silent* if and only if every execution is finite [12]. Let γ be a terminal configuration. The set of all possible executions starting from γ is the singleton $\{\gamma\}$. So, if $\mathcal{A}$ is self-stabilizing and silent, γ must be legitimate. Thus, to prove that a leader election algorithm is both self-stabilizing and silent, it is necessary and sufficient to show that: (1) in every terminal configuration γ, $\forall p, q \in V$, $Leader(p) = Leader(q)$ and $Leader(p)$ is the ID of some process; (2) every execution is finite.

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Algorithm $\mathcal{LE}$

In this chapter, we present a silent and self-stabilizing leader election algorithm, called $\mathcal{LE}$. Its formal code is given in Algorithm 1. Starting from an arbitrary configuration, $\mathcal{LE}$ converges to a terminal configuration, where the process of minimum ID, ℓ, is elected. More precisely, in the terminal configuration, every process p knows the identifier of ℓ thanks to its local variable $p.idR$; moreover a spanning tree rooted at ℓ is defined using two variables per process: par and $level$. First, $\ell.par = \ell$ and $\ell.level = 0$. Then, for every process $p \neq \ell$, $p.par$ points to the parent of p in the tree and $p.level$ is the level of p in the tree.

We present a simple algorithm for the leader election problem in Section 3.1. We show why this algorithm is not self-stabilizing in Section 3.2. Then, we explain in Section 3.3 how to modify this simple algorithm to make it self-stabilizing.

3.1 Non Self-Stabilizing Leader Election

We first consider a simplified version of $\mathcal{LE}$. Starting from a predefined initial configuration, it elects ℓ in all idR variables and builds a spanning tree rooted at ℓ.

Initially, every process p declares itself as leader: $p.idR = p$, $p.par = p$, and $p.level = 0$. So, p satisfies the two following predicates:

$$SelfRoot(p) \equiv (p.par = p) \text{ and } SelfRootOk'(p) \equiv (p.level = 0) \wedge (p.idR = p)$$

Note that, in the sequel, we say that p is a *self root* when $SelfRoot(p)$ holds.

From such an initial configuration, our non self-stabilizing algorithm consists in the following single action:

$$\begin{array}{llll} J\text{-}Action' & :: & \exists q \in \mathcal{N}_p, (q.idR < p.idR) & \to & p.par = \min_{\preceq}\{q \in \mathcal{N}_p\}; \\ & & & & p.idR = p.par.idR; \\ & & & & p.level = p.par.level + 1; \end{array}$$

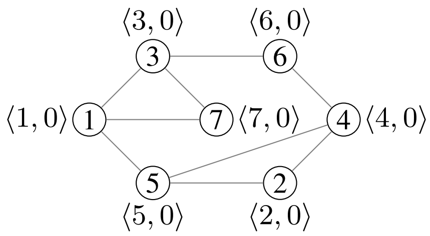
where $\forall x, y \in V, x \preceq y \Leftrightarrow (x.idR \leq y.idR) \wedge [(x.idR = y.idR) \Rightarrow (x < y)]$

Informally, when p discovers that $p.idR$ is not equal to the minimum identifier, it updates its variables accordingly: let q be the neighbor of p having idR minimal. Then, p selects q as new parent ($p.par = q$ and $p.level = p.par.level + 1$) and sets $p.idR$ to the value of $q.idR$. If there are several neighbors having idR minimal, we break ties using the identifiers of those neighbors.

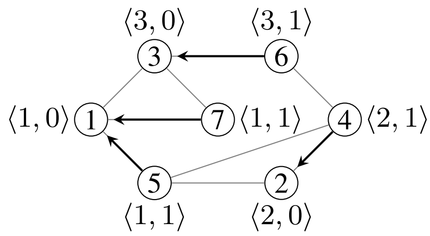
Hence, the identifier of ℓ is propagated, from neighbors to neighbors, into the idR variables and the system reaches a terminal configuration in $O(\mathcal{D})$ rounds. Figure 1 shows an example of such an execution.

Notice first that for every process p, $p.idR$ is always less than or equal to its own identifier. Indeed, $p.idR$ is initialized to p and decreases each time p executes $J\text{-}Action'$. Hence, $p.idR = p$ while p is a self root and after p executes $J\text{-}Action'$ for the first time, $p.idR$ is smaller than its ID forever.

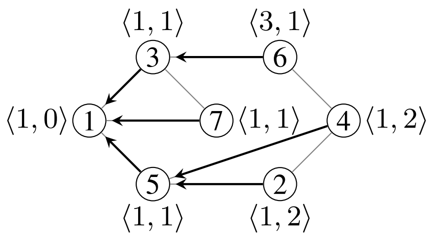
Second, even in this simplified context, for each two neighbors p and q such that q is the parent of p, it may happens that $p.idR$ is greater than $q.idR$—an example is shown in Figure 1c, where $p = 6$ and $q = 3$. This is due to the fact that p joins the tree of q but meanwhile q joins another tree and this change is not yet propagated to p. Similarly, when $p.idR \neq q.idR$,



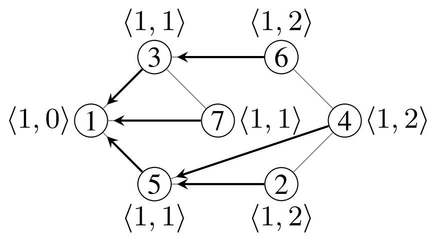
(a) Initial configuration. $SelfRoot(p) \wedge SelfRootOk'(p)$ holds for every process p.



(b) 4, 5, 6, and 7 have executed $J\text{-}Action'$. Note that $J\text{-}Action'$ was not enabled at 2 because it is a local minimum.



(c) 2, 3, and 4 have executed $J\text{-}Action'$. 3 joins the tree rooted at 1. However, the new value of $3.idR$ is not yet propagated to its child 6.



(d) 6 has executed $J\text{-}Action'$. The configuration is now terminal, $\ell = 1$ is elected, and a tree rooted at ℓ is available.

Figure 1: Example of execution of the non self-stabilizing algorithm. Process identifiers are given inside the nodes. $\langle x, y\rangle$ means $idR = x$ and $level = y$. Arrows represent par pointers. The absence of arrow means that the process is a self root.

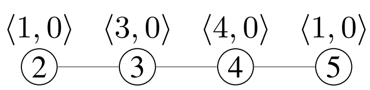
$p.level$ may be different from $q.level + 1$. According to those remarks, we can deduce that when $p.par = q$ with $q \neq p$, we have the following relation between p and q:

$$\begin{aligned} GoodIdR(p,q) &\equiv (p.idR \geq q.idR) \wedge (p.idR < p) \\ GoodLevel(p,q) &\equiv (p.idR = q.idR) \Rightarrow (p.level = q.level + 1) \end{aligned}$$

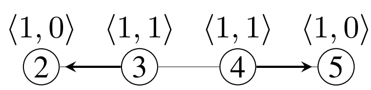
3.2 Fake IDs

This previous algorithm is not self-stabilizing. Indeed, in a self-stabilization context, the execution may start in an arbitrary configuration. In particular, idR variables can be initialized to arbitrary natural integer values, even values that are actually not IDs of (existing) processes. We call such values *fake IDs*.

The existence of fake IDs may lead the system to an illegitimate terminal configuration. Refer to the example of execution given in Figure 2: starting from Configuration 2a, if processes 3 and 4 move, the system reaches the terminal configuration given in 2b, where there are two trees and the idR variables elect the fake ID 1.



(a) Illegitimate initial configuration, where 2 and 5 have fake idR.



(b) 3 and 4 executed $J\text{-}Action'$. The configuration is now terminal.

Figure 2: Execution that does not converge to a legitimate configuration.

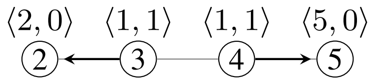


Figure 3: One step after Figure 2b, 2 and 5 have reset.

In this example, 2 and 5 can detect the problem. Indeed, predicate $SelfRootOk'$ is violated by both 2 and 5. One may believe that it is sufficient to reset the local state of 2 and 5 to $p.idR = p$, $p.par = p$ and $p.level = 0$. But, as shown on Figure 3 after their reset, there are still some errors. Again, 3 and 4 can detect the problem. Indeed, even if they are not self roots, predicate $GoodIdR(p, p.par) \wedge GoodLevel(p, p.par)$ is violated by both 3 and 4. So we define the following action:

$$\begin{array}{lll} R\text{-}Action' :: & \big(SelfRoot(p) \wedge \neg SelfRootOk'(p)\big) \vee \big(\neg SelfRoot(p) & \rightarrow \; p.par = p; \\ & \wedge \neg(GoodIdR(p, p.par) \wedge GoodLevel(p, p.par))\big) & \quad\; p.idR = p; \\ & & \quad\; p.level = 0; \end{array}$$

Unfortunately, this may lead to an execution that never converges, as shown in Figure 4. Indeed, if a process resets, it becomes a self root but this does not erase the fake ID in the rest of its subtree. Then, another process can join the tree and adopt the fake ID which will be further propagated, and so on. In the example, a process resets while another joins its tree at lower level, and this leads to endless erroneous behavior, since we do not want to assume any maximal value for $level$ (such an assumption would otherwise imply the knowledge of some upper bound on n). Therefore, the whole tree must be reset, instead of its root only. To that goal, we first froze the "abnormal" tree in order to forbid any process to join it, then the tree is reset top-down. The cleaning mechanism is detailed in the next section.

3.3 Cleaning Abnormal Trees

To introduce the trees, we define what is a "good relation" between a parent and its children. Namely, the predicate $KinshipOk'(p, q)$ models that a process p is a *real child* of its parent $q = p.par$. This predicate holds if and only if $GoodLevel(p, q)$ and $GoodIdR(p, q)$ are true. This relation defines a spanning forest: a *tree* is a maximal set of processes connected by par pointers and satisfying $KinshipOk'$ relation. A process p is a root of such a tree whenever $SelfRoot(p)$ holds or $KinshipOk'(p, p.par)$ is false. When $SelfRoot(p) \wedge SelfRootOk'(p)$ is true, p is a normal root just as in the non self-stabilizing case (see 3.1). In other cases, there is an error and p is said to be an *abnormal root*:

$$\begin{array}{lll} AbRoot'(p) & \equiv & \big(SelfRoot(p) \wedge \neg SelfRootOk'(p)\big) \\ & & \vee\big(\neg SelfRoot(p) \wedge \neg KinshipOk'(p, p.par)\big) \end{array}$$

These are the two possible errors identified in the Section 3.2. A tree is called an *abnormal tree* when its root is abnormal.

We now detail the different predicates and actions of Algorithm 1.

3.3.1 Variable $status$.

Abnormal trees need to be frozen before to be cleaned in order to prevent them from growing endlessly (see 3.2). This mechanism is achieved using an additional variable, $status$, that is used as follows. If a process is clean (*i.e.*, not involved into any freezing operation), then its $status$ is C. Otherwise, it has status EB or EF and no neighbor can select it as its parent. These two latter states are actually used to perform a "Propagation of Information with Feedback"

Algorithm 1 Algorithm $\mathcal{LE}$ for every process p

Variables

$p.idR \in \mathbb{N}, p.par \in \mathcal{N}_p \cup \{p\}, p.level \in \mathbb{N}, p.status \in \{C, EB, EF\}$

Macros

$$\begin{aligned}
Children_p &\equiv \{q \in \mathcal{N}_p \mid q.par = p\} \\
RealChildren_p &\equiv \{q \in Children_p \mid KinshipOk(q,p)\} \\
p \preceq q &\equiv (p.idR \le q.idR) \wedge [(p.idR = q.idR) \Rightarrow (p \le q)] \\
Min_p &\equiv \min_{\preceq} \{q \in \mathcal{N}_p \mid q.status = C\}
\end{aligned}$$

Predicates

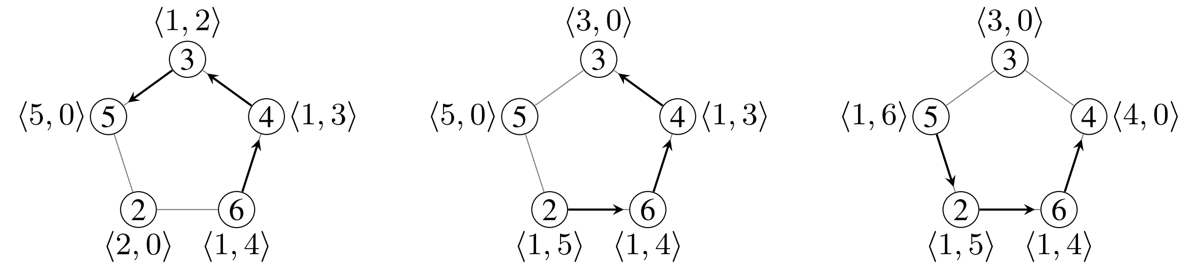
$$\begin{aligned}
SelfRoot(p) &\equiv p.par = p \\
SelfRootOk(p) &\equiv (p.level = 0) \wedge (p.idR = p) \wedge (p.status = C) \\
GoodIdR(s,f) &\equiv (s.idR \ge f.idR) \wedge (s.idR < s) \\
GoodLevel(s,f) &\equiv (s.idR = f.idR) \Rightarrow (s.level = f.level + 1) \\
GoodStatus(s,f) &\equiv [(s.status = EB) \Rightarrow (f.status = EB)] \\
&\qquad \vee [(s.status = EF) \Rightarrow (f.status \ne C)] \\
&\qquad \vee [(s.status = C) \Rightarrow (f.status \ne EF)] \\
KinshipOk(s,f) &\equiv GoodIdR(s,f) \wedge GoodLevel(s,f) \wedge GoodStatus(s,f) \\
AbRoot(p) &\equiv [SelfRoot(p) \wedge \neg SelfRootOk(p)] \\
&\qquad \vee [\neg SelfRoot(p) \wedge \neg KinshipOk(p, p.par)] \\
Allowed(p) &\equiv \forall q \in Children_p, (\neg KinshipOk(q,p) \Rightarrow q.status \ne C)
\end{aligned}$$

Guards

$$\begin{aligned}
EBroadcast(p) &\equiv (p.status = C) \wedge [AbRoot(p) \vee (p.par.status = EB)] \\
EFeedback(p) &\equiv (p.status = EB) \wedge (\forall q \in RealChildren_p, q.status = EF) \\
Reset(p) &\equiv (p.status = EF) \wedge AbRoot(p) \wedge Allowed(p) \\
Join(p) &\equiv (p.status = C) \wedge [\exists q \in \mathcal{N}_p, (q.idR < p.idR) \wedge (q.status = C)] \\
&\qquad \wedge Allowed(p)
\end{aligned}$$

Actions

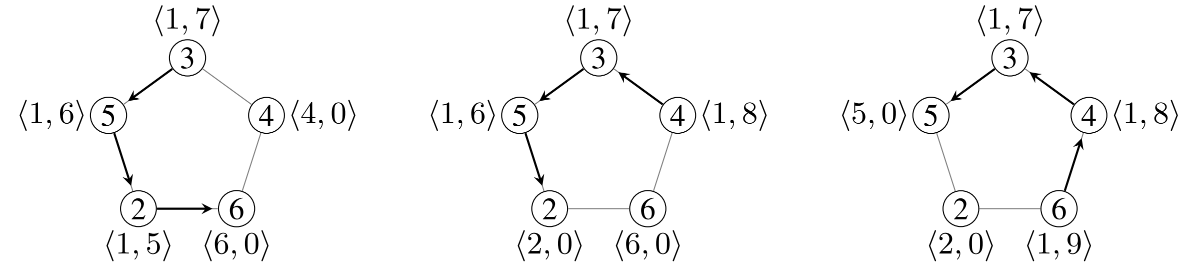
$$\begin{aligned}
&EB\text{-}action :: EBroadcast(p) \to p.status = EB; \\
&EF\text{-}action :: EFeedback(p) \to p.status = EF; \\
&R\text{-}action :: Reset(p) \to p.status = C;\ p.par = p; \\
&\qquad\qquad p.idR = p;\ p.level = 0; \\
&J\text{-}action :: Join(p) \wedge \neg EBroadcast(p) \to p.par = Min_p; \\
&\qquad\qquad p.idR = p.par.idR; \\
&\qquad\qquad p.level = p.par.level + 1;
\end{aligned}$$



(a) Illegitimate initial configuration.

(b) 2 joins the tree. 3 leaves it.

(c) 5 joins the tree. 4 leaves it.



(d) Both 3 and 6 move.

(e) 4 joins, 2 leaves.

(f) Configuration similar to 4a

Figure 4: The first process of the chain of bold arrows violates the predicate $SelfRootOk'$ and resets by executing $R\text{-}Action'$, while another process joins its tree. This cycle of resets and joins might never terminate.

[6, 16] into the abnormal trees. Therefore, status EB means "Error Broadcast" and EF means "Error Feedback". From an abnormal root, the status EB is broadcast down in the tree. Then, once the EB wave reaches a leaf, the leaf initiates a convergecast EF-wave. Once the EF-wave reaches the abnormal root, the tree is considered to be *dead*, meaning that there is no process of status C in the tree and no other process can join it. So, the tree can be safely reset from the abnormal root toward the leaves.

Notice that the new variable $status$ may also get arbitrary initialization. Thus, we enforce previously introduced predicates as follows.

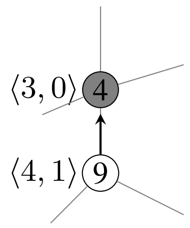
A self root must have status C, otherwise it is an abnormal root:

$$SelfRootOk(p) \equiv SelfRootOk'(p) \wedge (p.status = C)$$

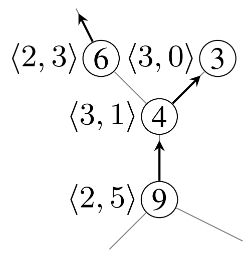
To be a real child of q, p should have a status coherent with the one of q. This is expressed with the predicate $GoodStatus(p, q)$, which is used to enforce the $KinshipOk(p, q)$ relation:

$$\begin{aligned} GoodStatus(p,q) &\equiv [(p.status = EB) \Rightarrow (q.status = EB)] \\ &\quad \vee[(p.status = EF) \Rightarrow (q.status \neq C)] \\ &\quad \vee[(p.status = C) \Rightarrow (q.status \neq EF)] \\ KinshipOk(p,q) &\equiv KinshipOk'(p,q) \wedge GoodStatus(p,q) \end{aligned}$$

Precisely, when p has status C, its parent must have status C or EB (if the EB-wave is not propagated yet to p). If p has status EB, its parent must be of status EB because p gets status EB from its parent and its parent will change its status to EF only after p gets status EF. Finally, if p has status EF, its parent can have status EB (if the EF-wave is not propagated yet to its parent) or EF.



(a) 4 and 9 are abnormal roots. If 4 executes R-$action$ before 9 executes EB-$action$, the kinship relation between 4 and 9 becomes correct and 9 is no more an abnormal root. Then, EB-$action$ is no more enabled at 9.



(b) 9 is an abnormal root and Min_4 is 6. If 4 executes J-$action$ before 9 executes EB-$action$, the kinship relation between 4 and 9 becomes correct and 9 is no more an abnormal root. Then, EB-$action$ is no more enabled at 9.

Figure 5: Example of situations where the parent of a process is locked.

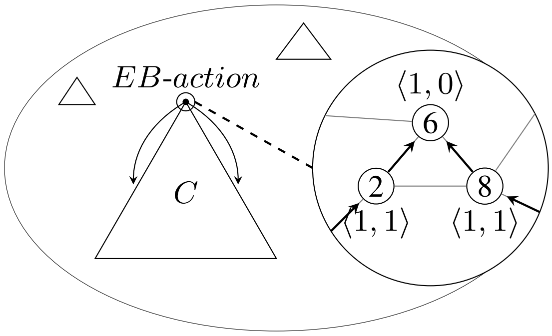
3.3.2 Normal Execution.

Remark that, after all abnormal trees have been removed, all processes have status C and the algorithm works as in the initial version. Notice that the guard of J-$action$ has been enforced so that only processes with status C and which are not abnormal root can execute it, and when executing J-$action$, a process can only choose a neighbor of status C as parent. Moreover, remark that the cleaning of all abnormal trees does not ensure that all fake IDs have been removed. Rather, it guarantees the removal of all fake IDs smaller than ℓ. This implies that (at least) ℓ is a self root at the end of the cleaning and all other processes will elect ℓ within the next $\mathcal{D}$ rounds.

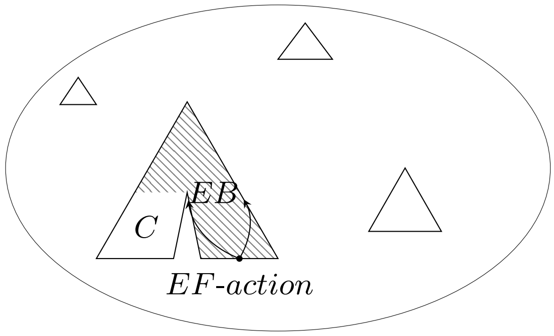
3.3.3 Cleaning Abnormal Trees.

Figure 6 shows how an abnormal tree is cleaned. In the first phase (see Figure 6a), the root broadcasts status EB down to its (abnormal) tree: all the processes in this tree execute EB-$action$, switch to status EB and are consequently informed that they are in an abnormal tree. The second phase starts when the EB-wave reaches a leaf. Then, a convergecast wave of status EF is initiated thanks to action EF-$action$ (see Figure 6b). The system is asynchronous, hence all the processes along some branch can have status EF before the broadcast of the EB-wave is done into another branch. In this case, the parent of these two branches waits that all its children in the tree (processes in the set $RealChildren$) get status EF before executing EF-$action$ (Figure 6c). When the root gets status EF, all processes have status EF: the tree is dead. Then (third phase), the root can reset (safely) to become a self root by executing R-$action$ (Figure 6e). Its former real children (of status EF) become themselves abnormal roots of dead trees (Figure 6f) and reset, *etc.*

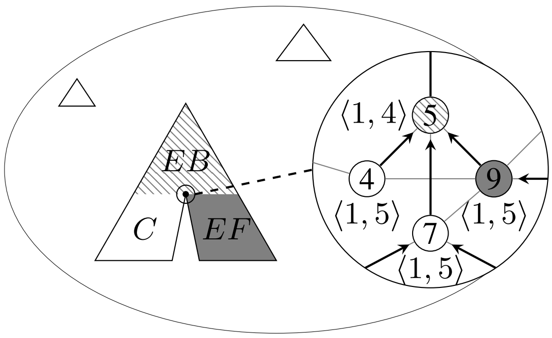
Finally, we used the predicate $Allowed(p)$ to temporarily lock the parent of p in two particular situations — illustrated in Figure 5 — where p is enabled to switch its status from C to EB. These locks impact neither the correctness nor the complexity of $\mathcal{LE}$. Rather, they allow us to simplify the proofs by ensuring that, once enabled, EB-$action$ remains continuously enabled until executed.



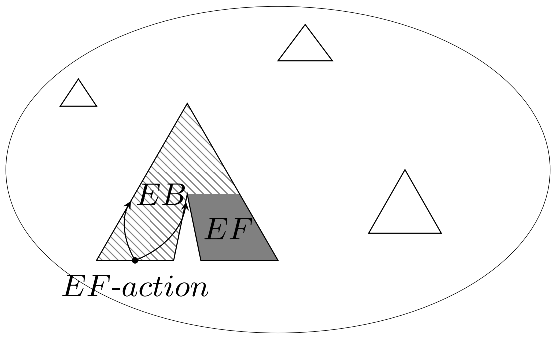
(a) When an abnormal root detects an error, it executes EB-$action$. The EB-wave is broadcast to the leaves. Here, 6 is an abnormal root because it is a self root and its idR is different from its ID ($1 \neq 6$).



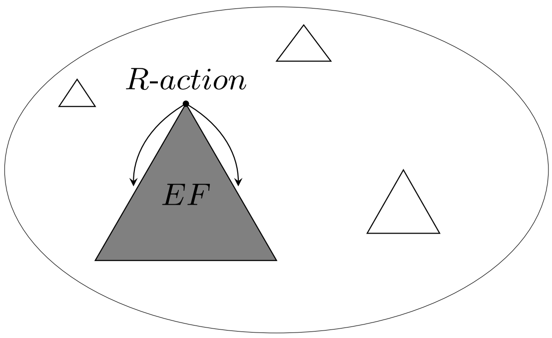
(b) When the EB-wave reaches a leaf, it executes EF-$action$. The EF-wave is propagated up to the root.



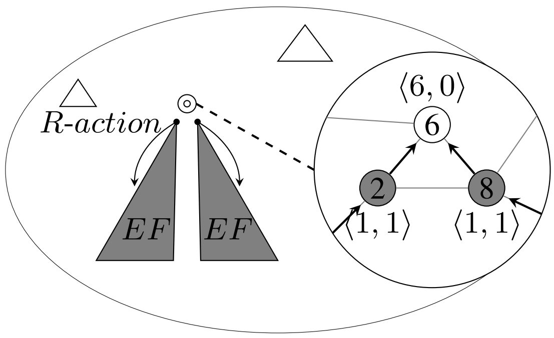
(c) It may happen that the EF-wave reaches a node, here process 5, even though the EB-wave is still broadcasting into some of its proper subtrees: 5 must wait that the status of 4 and 7 become EF before executing EF-$action$.



(d) EB-wave has been propagated in the other branch. An EF-wave is initiated by the leaves.



(e) EF-wave reaches the root. The root can safely reset (R-$action$) because its tree is dead. The cleaning wave is propagated down to the leaves.



(f) Its children become themselves abnormal roots of dead trees and can execute R-$action$: 2 and 8 can clean because their status is EF and their parent has status C.

Figure 6: Schematic example of the cleaning mechanism. Trees are filled according to the status of their processes: white for C, dashed for EB, gray for EF.

— 4 —

Correctness and Complexity Analysis

In this chapter, we first define some concepts which will be used in the proofs (Section 4.1). Then, we show in Section 4.2 that Algorithm $\mathcal{LE}$ is self-stabilizing and silent for the leader election, assuming a distributed unfair daemon. Along the proof, we also establish a bound on its stabilization time in steps, namely $O(n^3)$. Finally, we study more precisely the complexity of $\mathcal{LE}$ in Section 4.3 (in particular, we give its complexity in rounds).

4.1 Some definitions

First, we instantiate the function $Leader(p)$ used in the specification of the leader election (Section 2.5).

Definition 1 (Leader)**.** For each process p, for every configuration γ, the value $Leader(p)$ in γ is $p.idR$.

Note that the value of $Leader(p)$ depends on the current configuration γ. Nevertheless, when it is clear from the context, we omit the mention to γ. This will be also the case for every predicates and notations used in the sequel.

We now recall some definitions and notations from graph theory. A *path* $\mathcal{P}$, from p_k to p_0 is a sequence of processes $p_k, p_{k-1}, \ldots, p_0$ such that $p_{i-1} \in \mathcal{N}_{p_i}$, for all i in $\{1, ..., k\}$. Nodes p_k and p_0 are respectively called the *initial* and *terminal extremity* of $\mathcal{P}$. The length of $\mathcal{P}$, denoted by $|\mathcal{P}|$, is equal to k. We call *cycle* any path $p_k, p_{k-1}, \ldots, p_0$ such that $p_0 = p_k$. The distance between two processes p and q, denoted $\|p, q\|$, is equal to the length of the shortest path between p and q. The *diameter* of the network, denoted $\mathcal{D}$, is the maximum distance between any two processes.

The rest of the section is dedicated to introducing and justifying the notion of trees induced by the $KinshipOk$ relation. We first show that the predicate $KinshipOk$ is an acyclic relation. To that goal, we define the graph induced by the $KinshipOk$ relation.

Definition 2 (Kinship Relation Graph)**.** For some configuration γ, let $G_{kr} = (V, KR)$ be a directed graph such that $(p, q) \in KR \Leftrightarrow (\{p, q\} \in E) \wedge (p.par = q) \wedge KinshipOk(p, q)$. G_{kr} is called the graph of *kinship relations* in γ.

We first show that G_{kr} is a DAG (Directed Acyclic Graph). We recall, *path* and *cycle* naturally extend to directed graph, *i.e.*, a (directed) path $\mathcal{P}$ in G_{kr} is a sequence of processes $p_k, p_{k-1}, \ldots, p_0$ such that $(p_i, p_{i-1}) \in KR$, for all i in $\{1, ..., k\}$.

Lemma 1. *Let γ be a configuration. The graph of kinship relations in γ contains no cycle.*

Proof. By definition, for all pairs of processes p, q such that $KinshipOk(p, q)$ holds, we have: $p.idR \geq q.idR$ and $p.idR = q.idR \Rightarrow p.level = q.level + 1$. Hence, the processes along any path in G_{kr} are ordered w.r.t. the strict lexical order on the pair $(idR, level)$. The result directly follows. □

Hence G_{kr} is a DAG (Directed Acyclic Graph) and even a spanning forest since the condition $p.par = q$ implies at most one successor per process in KR. Below, we define the roots and trees of this spanning forest.

Definition 3 (Root)**.** For some configuration γ, a process p satisfies $Root(p)$ (and is called a *root* in γ) if and only if $SelfRoot(p) \vee AbRoot(p)$, or equivalently $SelfRoot(p) \vee \neg KinshipOk(p, p.par)$ holds in γ.

Next, we define the paths, called $KPath$s, that follow the tree structures in G_{kr}, *i.e.*, the paths linking each process to the root of its own tree.

Definition 4 (KPath)**.** For every process p, $KPath(p)$ is the unique path $p_0, p_1, \ldots, p_k$ such that $p_k = p$ and satisfying the following conditions:
– $\forall i, 1 \leq i \leq k, (p_i.par = p_{i-1}) \wedge KinshipOk(p_i, p_{i-1})$
– $Root(p_0)$

Using Definitions 3 and 4, we formally define trees as follows.

Definition 5 (Tree)**.** For some configuration γ, for every process p such that $Root(p)$, we define $Tree(p)$, the tree rooted at p, as follows:

$$Tree(p) = \{q \in V \mid p \text{ is the initial extremity of } KPath(q)\}$$

This means, in particular, that we identify each tree with the ID of its root.

We give in Observation 1 an invariant on KPaths when looking at the status of the processes. This property is based on the notion of S-Trace defined below.

Definition 6 (S-Trace)**.** For some configuration γ, for a sequence of processes $p_0, p_1, \ldots, p_k$, we define $S\text{-}Trace(p_0, p_1, \ldots, p_k) \in \{C, EB, EF\}^*$ as the sequence $(p_0.status).(p_1.status)$ $\ldots(p_k.status)$ in γ.

Observation 1. *For any configuration, we have:* $\forall p \in V, S\text{-}Trace(KPath(p)) \in EB^*C^* \cup EB^*EF^*$.

Proof. Let p be a process. If $|KPath(p)| = 1$, Observation 1 trivially holds. For $|KPath(p)| \geq 2$, assume by contradiction that $S\text{-}Trace(KPath(p)) \notin EB^*C^* \cup EB^*EF^*$. Then, $\exists s, f \in KPath(p)$ such that $s.par = f$ and $S\text{-}Trace(f, s) \in \{C.EB, C.EF, EF.EB, EF.C\}$. In all cases, $\neg GoodStatus(s, f)$ holds, which in turns implies that $\neg KinshipOk(s, f)$. This contradicts Definition 4. □

4.2 Correctness

To prove the self-stabilization of Algorithm $\mathcal{LE}$ under an unfair daemon, we first show that any execution is finite (Theorem 1) and then we show that in any terminal configuration, there is a unique leader: for every two processes, p and q, we have $Leader(p) = Leader(q)$ and $Leader(p)$ is the ID of some process (Theorem 2).

4.2.1 Termination of $\mathcal{LE}$

The goal, here, is to show that any execution contains a finite number of steps. We first partition a given execution into a finite number of segments (Lemma 4), see Fig. 7. Then, we prove that each segment contains a finite number of J-$actions$ (Lemma 10). This latter result implies that every execution contains a finite number of J-$actions$ (Corollary 2). Then, we show, in Lemma 11 and Corollary 3, that every execution contains a finite number of other actions. This allows us to conclude in Theorem 1 that every execution contains a finite number of steps.

Abnormal Trees.

First, we introduce some notions that refine the concept of trees.

Definition 7 (Normal/Abnormal Tree)**.** For every configuration γ and every process p, any tree rooted at p such that $\neg AbRoot(p)$ in γ is called a *normal tree*. In this case, $SelfRoot(p) \wedge SelfRootOk(p)$ holds in γ, by Definition 3. Any tree that is not normal is simply said to be *abnormal*.

Definition 8 (Alive/Dead)**.** Let γ be a configuration. A process p is called *alive* in γ if and only if $\gamma(p).status = C$. Otherwise, p is said to be *dead*. A tree T in γ is called an *alive tree* in γ if and only if $\exists p \in T$ such that p is alive in γ. Otherwise, it is called a *dead tree*.

Definition 9 (Leave/Join a Tree)**.** Let $\gamma \mapsto \gamma'$ be a step. If a process p is in a tree T in γ, but in a different tree T' in γ' (namely, the roots of T and T' are different), we say that p *leaves* T and *joins* T' in $\gamma \mapsto \gamma'$.

Remark 1. No process can join a dead tree.

Lemma 2. *No alive abnormal root can be created.*

Proof. Let p be a process which is not an alive abnormal root in some configuration γ. This means that p is dead, or p is a normal root ($SelfRoot(p) \wedge SelfRootOk(p)$ holds in γ), or p is not a root ($KinshipOk(p, p.par)$ holds in γ).

Let $\gamma \mapsto \gamma'$ be a step. If p executes EB-$action$ (respectively EF-$action$) during the step $\gamma \mapsto \gamma'$ then $\gamma'(p).status = EB$ (respectively $\gamma'(p).status = EF$) and, consequently, p is dead in γ'.

If p executes R-$action$, $SelfRoot(p) \wedge SelfRootOk(p)$ holds in γ'. So, p is a normal root in γ'.

If p executes J-$action$, let $q = Min_p$ in γ. By definition of J-$action$, $\gamma(q).status = C$, $\gamma(p).status = \gamma'(p).status = C$ and $\gamma(p).idR \leq p$ (since p is not an abnormal root at γ). Also, $\neg SelfRoot(p)$ holds in γ'.

- If q does not move in $\gamma \mapsto \gamma'$, then $\gamma'(p).par = q$, $\gamma'(q).status = C = \gamma'(p).status$, $\gamma'(p).level = \gamma(q).level + 1 = \gamma'(q).level + 1$, $\gamma'(p).idR = \gamma(q).idR = \gamma'(q).idR < \gamma(p).idR \leq p$. Hence, $KinshipOk(p, p.par)$ is true in γ'. Now, we already know that $\neg SelfRoot(p)$ holds in γ'. Thus, $\neg SelfRoot(p) \wedge KinshipOk(p, q)$ holds in γ': p is not a root in γ', by Definition 3.
- Assume now that q moves in $\gamma \mapsto \gamma'$. As $\gamma(q).status = C$, q can only execute EB-$action$ or J-$action$ in the step. Consequently, $\gamma'(q).idR \leq \gamma(q).idR$.
  Then, $\gamma'(p).idR = \gamma(q).idR \geq \gamma'(q).idR$ and $\gamma'(p).idR = \gamma(q).idR < \gamma(p).idR \leq p$. So, $GoodIdR(p, q)$ holds in γ'.

If q executes $J\text{-}action$, $\gamma'(p).idR \neq \gamma'(q).idR$. Otherwise, $\gamma'(p).idR = \gamma'(q).idR$ and $\gamma'(p).level = \gamma(q).level + 1 = \gamma'(q).level + 1$. So $GoodLevel(p, q)$ holds in γ'.
Finally, $\gamma'(p).status = \gamma(p).status = C$ and $\gamma'(q).status \in \{C, EB\}$, so the predicate $GoodStatus(p, q)$ holds in γ'.
Thus, $\neg SelfRoot(p) \wedge KinshipOk(p, q)$ holds in γ' and, so, p is not a root in γ', by Definition 3.

Assume now that p executes no action in the step $\gamma \mapsto \gamma'$. The only way for p to become an alive abnormal root is that $\gamma(p).par$ moves during the step, since the property "alive abnormal root" only depends on p and $p.par$. Furthermore, as p is not an alive abnormal root, when p is a normal root in γ, it stays so, in γ'.

Therefore, let us consider the case when p is not a root in γ and $\gamma(p).par$ moves. As p changes none of its variables, the only way for it to become an alive abnormal root is to have status C in γ and thus in γ'. As $GoodStatus(p, p.par)$ holds in γ, this implies that the status of $\gamma(p).par$ is either EB or C. Looking at case EB, p is a real child of $p.par$ in γ with status C; hence $EF\text{-}action$ is disabled for $p.par$ in γ. Looking at case C, $p.par$ can execute $EB\text{-}action$ and can change only its status to EB in $\gamma \mapsto \gamma'$: $GoodStatus(p, p.par)$ holds in γ' and consequently $KinshipOk(p, p.par)$ holds in γ'. $p.par$ can also execute $J\text{-}action$ in $\gamma \mapsto \gamma'$. This means that in γ and γ', $p.par$ has status C, hence $GoodStatus(p, p.par)$ holds in γ'. Furthermore, $p.par$ has a smaller value of idR in γ', hence $GoodIdR(p, p.par)$ and $GoodLevel(p, p.par)$ are satisfied in γ', and consequently $KinshipOk(p, p.par)$ holds in γ'. □

Lemma 3. *No alive abnormal tree can be created.*

Proof. Let $\gamma \mapsto \gamma'$ a step. Let $p \in V$. Assume there is no alive abnormal tree rooted at p in γ. In particular, p is not an alive abnormal root in γ. Then, assume, by contradiction, that $Tree(p)$ exists and is an alive abnormal tree in γ'.

- If $\gamma'(p).status = EF$, then every process in the tree has status EF (Observation 1) and the tree is dead, a contradiction.
- If $\gamma'(p).status = C$, then p is an alive abnormal root in γ'. But no alive abnormal root is created (Lemma 2), a contradiction.
- If $\gamma'(p).status = EB$. Then, according to the algorithm, there are two possible cases:

  $\gamma(p).status = EB$**:**
  - If $AbRoot(p)$ holds in γ, then $Tree(p)$ is dead in γ (otherwise, $Tree(p)$ is an abnormal alive tree in γ, a contradiction). By the definition of $J\text{-}action$, no process can join $Tree(p)$ in $\gamma \mapsto \gamma'$. Moreover, as $\gamma(p).status = EB$, no process q in $Tree(p)$ satisfies $Reset(q)$ in γ, by Observation 1. Consequently, no process can leave $Tree(p)$ in $\gamma \mapsto \gamma'$. So, every process in $Tree(p)$ still have status EF or EB in γ', *i.e.* $Tree(p)$ is still dead in γ', a contradiction.
  - If $\neg AbRoot(p)$ holds in γ, then p does not satisfy the predicate $SelfRoot(p)$, otherwise $SelfRootOk(p)$ implies that $\gamma(p).status = C$, a contradiction. So, let $q = \gamma(p).par \in \mathcal{N}_p$. $\neg AbRoot(p)$ in γ implies that $q.status = EB$ and $KinshipOk(p, q)$ in γ. This latter also implies that $p \in RealChildren_q$ in γ. Now, $p \in RealChildren_q$ and $p.status = EB$ in γ implies that q is disabled in γ. Moreover, as $\gamma'(p).status = EB$, p does not execute any action in $\gamma \mapsto \gamma'$. So, $\neg AbRoot(p)$ still holds in γ', a contradiction.

  $\gamma(p).status = C$**:** Then, $\neg AbRoot(p)$ holds in γ (otherwise p is an abnormal alive root in γ). Then, p executes $EB\text{-}action$ in $\gamma \mapsto \gamma'$ to get status EB. So, $EBroadcast(p) \wedge$

$\neg AbRoot(p)$ implies that $p.par \neq p$ and $p.par.status = EB$ in γ. So, let $q = \gamma(p).par \in \mathcal{N}_p$. Now $p.par \neq p \wedge \neg AbRoot(p)$ implies that $KinshipOk(p, q)$ in γ. So, $p \in RealChildren_q$ and, as $p.status = C$ and $q.status = EB$ in γ, q is disabled in γ. Moreover, as $\gamma'(p).status = EB$, p necessarily executes EB-*action* in $\gamma \mapsto \gamma'$, which only changes its status to EB. So, $\neg AbRoot(p)$ still holds in γ', a contradiction. □

Finite Number of J-*action*s.

To show that every process p executes only a finite number of J-*action*s, we prove below that p can only execute a finite number of J-*action*s in each segment of execution — a segment being separated from its follower by the death or the disappearance of some tree.

Definition 10 (Disappear/Die)**.** Let $\gamma \mapsto \gamma'$ be some step and let p be a process such that $Root(p)$ in γ.

$Tree(p)$ *disappears* during the step $\gamma \mapsto \gamma'$ if and only if $Tree(p)$ is no more defined in γ' — namely $Root(p)$ does not hold in γ'.

$Tree(p)$ *dies* during the step $\gamma \mapsto \gamma'$ if and only if $Tree(p)$ is alive in γ, yet $Tree(p)$ exists — namely $Root(p)$ holds — and is dead in γ'.

Definition 11 (Segment of execution)**.** Let $e = \gamma_0\gamma_1 \ldots$ be any execution. $e' = \gamma_i \ldots \gamma_j$ is a *segment of execution* e (segment, for short) if and only if e' is a maximal factor of e, where no tree dies nor disappears.

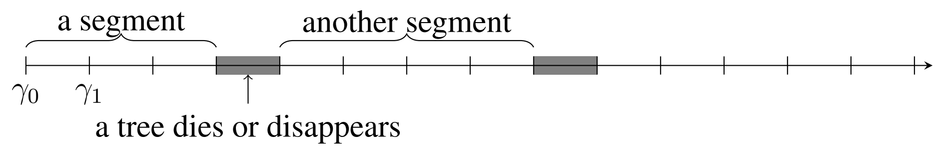


Figure 7: Segments of execution

Figure 7 illustrates Definition 11. We now show that the number of segments is finite.

Lemma 4. *There are at most $n + 1$ segments in any execution.*

Proof. In the initial configuration, there are at most n abnormal roots (every process) and, consequently, at most n abnormal trees. As no alive abnormal tree can be created (Lemma 3), if an abnormal tree is alive, then it is alive since the initial configuration. So, there is at most n trees that die or disappear and, consequently, there are at most $n + 1$ segments in the execution. □

We now count the number of J-*action*s processes can execute in a given segment. For that purpose, we first need to prove intermediate lemmas that identify properties on computation steps.

Observation 2. *Let γ be a configuration and let p a process such that $Reset(p)$ is true in γ. Then, $Tree(p)$ exists and is dead in γ.*

Proof. Let γ be a configuration and let p be a process such that $Reset(p)$ is true in γ. By definition, $AbRoot(p)$ holds in γ, hence $Tree(p)$ is defined in γ. Furthermore, $\gamma(p).status = EF$: by Observation 1, every process in $Tree(p)$ has status EF in γ, and we are done. □

Lemma 5. *Let $\gamma \mapsto \gamma'$ be a step and let p be a process such that $p.status \in \{EB, EF\}$ in γ. Let T be the which contains p in γ. First, T is an abnormal tree in Second, if T does not disappear during the step $\gamma \mapsto \gamma'$, p is still in T in γ' unless T was dead in γ.*

Proof. Let $\gamma \mapsto \gamma'$ be a step and let p be a process such that $p.status \in \{EB, EF\}$ in γ. We note r the root of the tree containing p in γ. As $S\text{-}Trace(KPath(p)) \in EB^*EF^*$, by Observation 1, the status of r in γ is either EF or EB. Hence $AbRoot(r)$ holds in γ: $Tree(r)$ is an abnormal tree in γ.

Assume now that $Root(r)$ holds in γ' (the tree does not disappear during the step). If r executes $R\text{-}action$ in $\gamma \mapsto \gamma'$, Observation 2 applies in γ and proves that $Tree(r)$ is dead in γ.

If r does not (or cannot) execute $R\text{-}action$, its only possible action is $EF\text{-}action$. As $Root(r)$ holds in γ', r is still abnormal root in γ'. Let then $q \in KPath(p)$ in γ with $q \neq r$. By Observation 1, $\gamma(q).status \in \{EB, EF\}$ also. If $\gamma(q).status = EB$, q can only execute $EF\text{-}action$ and if $\gamma(q).status = EF$, q is disabled, as $q \neq r$. Executing $EF\text{-}action$ preserves $GoodStatus$ and hence $KinshipOk$ relations. Therefore, the $KPath$ from p to r is the same in γ and γ' and then $p \in Tree(r)$ in γ'. □

Lemma 6. *Let p be a process and let $\gamma \mapsto \gamma'$ be step. If p is an abnormal root of status C in γ, then it is still an abnormal root in γ'.*

Proof. Let $\gamma \mapsto \gamma'$ be step and let p be a process such that $AbRoot(p) \wedge p.status = C$ in γ: p can only execute $EB\text{-}action$. Therefore, $\gamma'(p).status \in \{C, EB\}$ and every other variable of p has identical value in γ and γ'.

So, if $SelfRoot(p)$ holds in γ, then $SelfRootOk(p)$ is false in γ, and $SelfRoot(p) \wedge \neg SelfRootOk(p)$ still holds in γ'.

Otherwise, $\neg SelfRoot(p)$ holds in γ, *i.e.*, $p.par \neq p$. Then, $\neg SelfRoot(p)$ still holds in γ'. Let $q = \gamma(p).par$ and consider the following cases:

$\gamma(q).status = EF$**:** Then, $\neg GoodStatus(p, q)$ holds in γ, which implies $\neg KinshipOk(p, q)$ holds in γ. However, $p \in Children_q$ in γ. So, $\neg Allowed(q)$ holds in γ, and q is disabled. So, $\gamma'(p).status \in \{C, EB\}$ and $\gamma'(q).status = EF$, which implies $\neg GoodStatus(p, q)$ in γ'. Thus, $\neg KinshipOk(p, q)$ holds in γ'.

$\gamma(q).status = EB$**:** Then, $GoodStatus(p, q)$ holds in γ. So, $AbRoot(p)$ in γ implies that $\neg GoodIdR(p, q) \vee \neg GoodLevel(p, q)$ holds in γ. Now, q can only executes $EF\text{-}action$ in $\gamma \mapsto \gamma'$. So, neither p nor q modify their variables par, idR, or $level$ in $\gamma \mapsto \gamma'$, and, consequently, $\neg GoodIdR(p, q) \vee \neg GoodLevel(p, q)$ still holds in γ'. So, $\neg KinshipOk(p, q)$ holds in γ'.

$\gamma(q).status = C$**:** $AbRoot(p)$ in γ implies that $\neg KinshipOk(p, q)$ holds in γ. Thus, $\neg Allowed(q)$ holds in γ because $p \in Children_q$ and $p.status = C$ in γ. So, q cannot execute $J\text{-}action$ in $\gamma \mapsto \gamma'$.

Then, as $\gamma(q).status = C \wedge \gamma(p).status = C$, $GoodStatus(p, q)$ holds in γ. So, $AbRoot(p)$ in γ implies that $\neg GoodIdR(p, q) \vee \neg GoodLevel(p, q)$ holds in γ. As p and q can only modify their status in $\gamma \mapsto \gamma'$ (q can only execute $EB\text{-}action$ in $\gamma \mapsto \gamma'$), $\neg GoodIdR(p, q) \vee \neg GoodLevel(p, q)$ still holds in γ'. So, $\neg KinshipOk(p, q)$ holds in γ'.

In any cases, $\neg KinshipOk(p, q)$ holds in γ'. As $\neg SelfRoot(p)$ holds in γ', $AbRoot(p)$ holds in γ'. □

Lemma 7. *Let γ be a configuration and let p be a process such that $p.status \in \{EB, EF\}$ in γ. Let T be the tree which contains p in γ. Let γ_R be the first configuration, if any, after γ, such that p executes an R-action $\gamma_R \mapsto \gamma_{R+1}$.*

Assume γ_R exists, then T is dead in γ_R or has disappeared (at least once) between γ and γ_R.

Proof. Let γ be a configuration and let p be a process such that $p.status \in \{EB, EF\}$ in γ. We note r the root of the tree which contains p in γ. Let $\gamma = \gamma_0\gamma_1...$ be an execution starting at γ. Let γ_R be the first configuration, if any, in this execution such that p executes an $R\text{-}action$ during the step $\gamma_R \mapsto \gamma_{R+1}$.

For every configuration γ_x, $x \in \{0, ..., R-1\}$, the status of p is EB or EF. Hence, Lemma 5 applies iteratively in γ_x: either $Tree(r)$ disappears during the step $\gamma_x \mapsto \gamma_{x+1}$, or, if not, $p \in Tree(r)$ in γ_{x+1}. Hence, in γ_R, either $Tree(r)$ has disappeared or, if not, $p \in Tree(r)$.

When $p \in Tree(r)$ in γ_R, by assumption, p executes an $R\text{-}action$ between γ_R and γ_{R+1}. Hence, $AbRoot(p)$ holds in γ_R and thus $p = r$. Furthermore, Observation 2 applies and proves that $Tree(r)$ is dead in γ_R. □

Lemma 8. *Let p be a process and let $\gamma \mapsto \gamma'$ be a step. Let T be the tree which contains p in γ. If $EBroadcast(p)$ holds in γ, then T is an abnormal alive tree in γ and if T has not disappeared in γ', p still belongs to T in γ'.*

Proof. Let $\gamma \mapsto \gamma'$ be a step. Let $p \in V$ such that $EBroadcast(p)$ holds in γ. We note r the root of the tree which contains p in γ.

If $AbRoot(p)$ holds in γ, then $p = r$ is the root of an alive abnormal tree, since $\gamma(p).status = C$. Furthermore, if $Tree(p)$ exists in γ', $p \in Tree(p)$ in γ', trivially.

Otherwise, $\neg AbRoot(p)$, $p.par.status = EB$, and $KinshipOk(p, p.par)$ holds in γ. Applying Lemma 5 to $\gamma(p).par$, we have that $\gamma(p).par$ belongs to an abnormal alive tree in γ and so does p: $Tree(r)$ is an alive abnormal tree.

Furthermore, first note that $\gamma(p).par = \gamma'(p).par$ (p can only change its status to EB in $\gamma \mapsto \gamma'$: either p do not move or executes $EB\text{-}action$). So, still by Lemma 5, in γ', if $Tree(r)$ exists in γ', $\gamma'(p).par$ belongs to $Tree(r)$ in γ', since $Tree(r)$ is not dead in γ ($\gamma(p).status = C$). As $KinshipOk(p, p.par)$ holds in γ, we have that $p \in RealChildren_q$ in γ. Since $\gamma(p).status = C$, q is disabled in γ (because of p) and, as p can only modify its status to EB in $\gamma \mapsto \gamma'$, we still have $p \in RealChildren_q$ in γ', *i.e.*, p and q belong to the same abnormal tree, $Tree(r)$, in γ'. □

Corollary 1. *Let γ be a configuration and let p be a process such that $EBroadcast(p)$ holds in γ. Let T the tree which contains p in γ. Let γ_R be the first configuration, if any, since γ, such that p executes an R-action $\gamma_R \mapsto \gamma_{R+1}$.*

Assume γ_R exists, then T is an alive abnormal tree in γ but it is dead in γ_R or has disappeared (at least once) between γ and γ_R.

Proof. Let γ be a configuration and let p be a process such that $EBroadcast(p)$ holds in γ. We note r the root of the tree which contains p in γ. Lemma 8 applies in γ: $Tree(r)$ is an alive abnormal tree in γ.

Let $\gamma = \gamma_0\gamma_1...$ be an execution starting at γ. Let γ_R be the first configuration, if any, in this execution such that p executes an $R\text{-}action$ during the step $\gamma_R \mapsto \gamma_{R+1}$. We assume that γ_R exists. Then at some step, $\gamma_i \mapsto \gamma_{i+1}$, p executes a $EB\text{-}action$, with $i < R$.

Lemma 8 applies iteratively from γ_0 and to γ_i: either $Tree(r)$ has disappeared in γ_1 (and so between γ_0 and γ_{i+1}), or p stays in $Tree(r)$ in γ_1 (and so between γ_0 and γ_{i+1}), and so on.

If $Tree(r)$ has not yet disappeared in γ_{i+1}, $p \in Tree(r)$ in γ_{i+1} with $\gamma_{i+1}(p).status = EB$. Here, Lemma 7 applies and proves that $Tree(r)$ has disappeared in γ_R or p is in $Tree(r)$ in γ_R. □

Lemma 9. *Let p be a process. Let s be a segment of execution. Between any two executions of J-$action$ by p in s, p can only execute J-actions.*

Proof. Let $s = \gamma_0\gamma_1 \dots$ be a segment of execution and $p \in V$. Consider two executions of J-$action$ by p during s: one in $\gamma_i \mapsto \gamma_{i+1}$ and the other in $\gamma_j \mapsto \gamma_{j+1}$, with $i < j$. Assume by contradiction that p executes an action different from J-$action$ between γ_{i+1} and γ_j. Let $\gamma_k \mapsto \gamma_{k+1}$ be the first step between γ_{i+1} and γ_j during which p executes some other action: this is a EB-$action$. Let $\gamma_l \mapsto \gamma_{l+1}$ be the last step between γ_{i+1} and γ_j during which p executes some other action: this is a R-$action$ (hence $k < l$).

Now, Lemma 1 applies since in γ_k, $EBroadcast(p)$ holds, and in some step later $\gamma_l \mapsto \gamma_{l+1}$, p executes a R-$action$. This proves that in γ_k, some abnormal tree is alive and that in γ_l, this tree is dead or has disappeared. Hence γ_k and γ_l are not in the same segment, a contradiction. □

Lemma 10. *In a segment of execution, there are at most $(n-1)(n-2)/2$ executions of J-$action$.*

Proof. Let $p \in V$. First, p only executes J-$action$s between two J-$action$s in the same segment (Lemma 9). So, using the guard of J-$action$, it follows that the value of the $p.idR$ always decreases during any sequence of J-$action$s, which means that p cannot set $p.idR$ two times to the same value during the segment.

Let A be the set of processes q such that $q.status = C$ at the beginning of the segment. Let B the set of processes q such that q executes an R-$action$ in the segment. $A \cap B = \emptyset$. Indeed, pick a process $q \in A \cap B$. q switches from status C at the beginning to status EB, and then to status EF since some step later, it executes R-$action$. Hence, there exists a configuration γ_b in the segment such that $EBroadcast(q)$ is true and another γ_r, later on such that R-$action$ occurs: hence Corollary 1 applies and proves that the tree of q in γ_b is abnormal alive and that it dies or disappears some step before γ_r. This contradicts the definition of segment. Hence, $|A| + |B| \leq n$.

Now, $p.idR$ can only get values from the idR of processes in A or from the ID of processes in B. Let $f : V \mapsto \mathbb{N}$ such that $\forall p \in A \cup B$, if $p \in A$, $f(p) = x$, where x is the value of $p.idR$ at the beginning of the segment; otherwise, $f(p) = p$. Let $p_0, \dots p_{k-1}$ (with $k \leq n$) be the set of processes in $A \cup B$ in ascending order of f. p_i changes at most i times of idR. Hence, in a given segment, the number of executed J-$action$s, noted $\sharp J$-$action$, satisfies the following inequality:

$$\sharp J\text{-}action \leq \sum_{i=0}^{k-1} i \leq \sum_{i=0}^{n-1} i = \frac{(n-1)(n-2)}{2}$$

□

By Lemmas 4 and 10, in any execution, there are at most $n+1$ segments, where processes execute at most $(n-1)(n-2)/2$ J-$action$s. Moreover, by definition, there are at most n steps outside segments (more precisely, the steps where at least one abnormal tree dies or disappears) where some J-$action$s may be executed. Hence, follows:

Corollary 2. *In any execution, there are at most $\frac{n^3}{2} - n^2 + \frac{n}{2} + 1$ steps containing J-actions.*

Other Actions.

Below, we show an upper bound on the number of executions of other actions.

Lemma 11. *In any execution, each process can execute at most n R-actions.*

Proof. First, by definition, there are at most n abnormal alive trees in the initial configuration. Let $\sharp AbT$ be that number. Moreover, $\sharp AbT$ can only decrease, by Lemma 3.

Let p be a process. We first show that when p executes $R\text{-}action$ for the first time, $\sharp AbT \leq n-1$. Then, we show that after every subsequent execution of $R\text{-}action$ by p, $\sharp AbT$ necessarily decreases. Hence, we will conclude that p cannot execute $R\text{-}action$ more than n, because $\sharp AbT$ cannot be negative.

Consider the first step $\gamma_i \mapsto \gamma_{i+1}$ where p executes $R\text{-}action$. Using Observation 2, $Tree(p)$ exists and is dead in γ_i. Hence, there are at most $n-1$ abnormal alive trees in γ_i.

Consider the $j-th$ execution of $R\text{-}action$ by p, with $j > 1$. After the $(j-1)-th$ $R\text{-}action$ of p, the status of p is C. So, between the $(j-1)-th$ and the $j-th$ $R\text{-}action$, the status of p thus switches from C to EB and from C to EF, so that p can switch its status from EF to C when executing its $j-th$ $R\text{-}action$. Hence, meanwhile there exists a configuration γ_b such that $EBroadcast(q)$ is true and another γ_r, later on such that p executes its $j-th$ $R\text{-}action$ in $\gamma_r \mapsto \gamma_{r+1}$: Corollary 1 applies and proves that the tree to which p belongs in γ_b is abnormal alive and that tree dies or disappears some step before γ_r, and we are done. □

Let p be a process. p necessarily executes $R\text{-}action$ between two executions of $EF\text{-}action$ (resp. $EB\text{-}action$). Hence, we have the following corollary.

Corollary 3. *In any execution, a process can execute $EB\text{-}action$ and $EF\text{-}action$ at most n times, each.*

By Corollaries 2, 3, and Lemma 11:

Theorem 1 (Convergence)**.** Every execution contains at most $\frac{n^3}{2} + 2n^2 + \frac{n}{2} + 1$ steps.

4.2.2 Terminal Configurations

We now show that in a terminal configuration, there is one and only one leader process, known by all processes, *i.e.*, for every two processes, p and q, we have $Leader(p) = Leader(q)$ and $Leader(p)$ is the ID of some process.

Lemma 12. *In a terminal configuration, every process has status C.*

Proof. By contradiction, consider a terminal configuration γ where some process p satisfies $p.status \neq C$. Then two cases are possible:

1. $p.status = EB$. By Observation 1, $\exists q \in V$ such that $q.status = EB \wedge (\forall q' \in RealChildren_q, q'.status \neq EB) \wedge p \in KPath(q)$. If $RealChildren_q = \emptyset$, then q can executes $EF\text{-}action$. Otherwise, there are two cases. Either $\forall q' \in RealChildren_q$, $q'.status = EF$ and q can execute $EF\text{-}action$, or $\exists q' \in RealChildren_q$, $q'.status = C$ then q' can execute $EB\text{-}action$. Hence, in both cases, γ is not terminal, a contradiction.
2. $p.status = EF$. By Observation 1, $\exists q \in V$ such that $q.status = EF \wedge (Root(q) \vee (KinshipOk(q, q.par) \wedge q.par.status \neq EF) \wedge q \in KPath(p)$.

   If $Root(q)$, then $AbRoot(q) \vee SelfRoot(q)$. Now, $q.status = EF$ implies that $AbRoot(q)$ holds. So, in all cases, $q.status = EF \wedge AbRoot(q)$ holds. If $Allowed(q)$ holds, then

$R\text{-}action$ is enabled at q, a contradiction. Otherwise, $\exists r \in Children_q, \neg KinshipOk(r, q) \wedge r.status = C$. In this case, $EB\text{-}action$ is enabled at r, a contradiction.

If $\neg Root(q)$, then there are two cases. Either $q.par.status = C$, $AbRoot(q)$ holds and we obtain a contradiction as in the case where $Root(q)$ holds. Or,$q.par.status = EB$ and using the same argument as in case 1, we can deduce that some process is enabled, a contradiction.

Hence, all cases, γ is not terminal, a contradiction. □

Theorem 2 (Correctness)**.** In a terminal configuration, $\forall p, q \in V, Leader(p) = Leader(q)$ and $Leader(p)$ is the ID of some process.

Proof. Let γ a terminal configuration. Assume first, by contradiction, that there are at least two leaders. Then, G being connected, $\exists p, q \in V$ such that $Leader(\gamma(p)) \neq Leader(\gamma(q))$ and $q \in \mathcal{N}_p$. Assume without loss of generality that $Leader(\gamma(p)) = \gamma(p).idR < \gamma(q).idR = Leader(\gamma(q))$. By Lemma 12, $p.status = q.status = C$. Then, either $EBroadcast(q)$ is true and q can execute $EB\text{-}action$ or q can execute $J\text{-}action$. Hence γ is not terminal, a contradiction.

Assume now that the leader is not one of the processes, *i.e.*, is a fake ID. Let $p \in V$ such that its $level$ is minimum. Notice that $\gamma(p).status = C$ by Lemma 12. If $SelfRoot(p)$ holds in γ, $\gamma(p).idR \neq p$. So, $AbRoot(p)$ holds and p can execute $EB\text{-}action$. Otherwise, there is $q \in \mathcal{N}_p$ such that $\gamma(p).par = q$. The level of p being minimum, we have $\gamma(p).level \leq \gamma(q).level$. So, $AbRoot(p)$ holds and p can execute $EB\text{-}action$. Hence, γ is not terminal, a contradiction. □

Using Theorem 2, there is exactly one root in a terminal configuration (the leader elected). So the graph of kinship relations in a terminal configuration contains exactly one tree. Hence, we can conclude:

Remark 2. In a terminal configuration, G_{kr} is a spanning tree rooted at the leader.

Theorems 1 and 2 establish the self-stabilization, silence, and step complexity of Algorithm $\mathcal{LE}$. Moreover, note that idR and $level$ can be stored in $\Theta(\log n)$ bits. Hence, we can conclude:

Theorem 3. Algorithm $\mathcal{LE}$ is a self-stabilizing and silent leader election algorithm working under a distributed unfair daemon. Its step complexity is at most $\frac{n^3}{2} + 2n^2 + \frac{n}{2} + 1$ steps. Its memory requirement is $\Theta(\log n)$ bits per process.

4.3 Complexity Analysis

In this section, we study the complexity of Algorithm $\mathcal{LE}$ in rounds and we make a worst-case analysis of its stabilization time both in steps and rounds.

4.3.1 Stabilization Time in Rounds

Clean Configurations.

First, we study the "good" cases, *i.e.*, when the system is in a clean configuration (defined below). From such configurations, the execution consists in building a tree rooted at ℓ using $J\text{-}action$ only. Once, the tree is built, the system is in a terminal configuration, where every process has elected ℓ.

Definition 12 (Clean configuration)**.** A configuration γ is called a *clean configuration* if and only if for every process p, $\neg EBroadcast(p) \wedge p.status = C$ holds in γ. A configuration that is not clean is said to be *dirty*.

Remark 3. By definition, in a clean configuration, every process p has status C and either p is a normal root, *i.e.*, $SelfRoot(p) \wedge SelfRootOk(p)$, or (exclusively) $KinshipOk(p, p.par)$ holds.

Remark 4. Notice that in a clean configuration, the only action a process p can execute is J-$action$, provided that $Join(p)$ holds. Note also that $Allowed(p)$ always holds due to Remark 3. Verifying $Join(p)$ then reduces to: $\exists q \in \mathcal{N}_p, (q.idR < p.idR)$. In this case, the value of $p.idR$ can only decrease.

Lemmas 13 to 16 proves that, starting from a clean configuration, the system reaches in $O(\mathcal{D})$ rounds a terminal configuration (see Theorem 4). We first show the set of clean configurations is closed.

Lemma 13. *The set of clean configurations is closed.*

Proof. Let $\gamma \mapsto \gamma'$ be a step such that γ is a clean configuration. By definition, all processes have status C in γ. So, processes can only execute J-$action$ (Remark 4) in $\gamma \mapsto \gamma'$, and consequently all processes have status C in γ'. Now, $\forall p \in V, \neg EBroadcast(p) \wedge p.status = C$ in γ implies that there is no alive abnormal root in γ. By Lemma 2, there is no alive abnormal root in γ' too. Now, the fact that all processes have status C and there is no alive abnormal root in γ' implies that $\forall p \in V, \neg EBroadcast(p) \wedge p.status = C$ in γ', *i.e.*, γ' is clean. □

Using Lemma 13, we show below that if a process is enabled in a clean configuration — for the only action it can execute, *i.e.*, J-$action$ — it remains enabled until it executes it.

Lemma 14. *In a clean configuration, if J-$action$ is enabled at p, it remains enabled until it is executed by p.*

Proof. Let $\gamma \mapsto \gamma'$ be a step such that γ is a clean configuration. Assume by contradiction that J-$action$ is enabled at p in γ and not in γ', but p did not execute J-$action$ between γ and γ'. By Lemma 13, γ' is also a clean configuration. So, $\neg EBroadcast(p) \wedge p.status = C$ holds in γ'.

But $Join(p)$ must be false in γ'. Using Remark 4, this means that there necessarily exists a neighbor of p, say q, such that $\gamma(q).idR < \gamma(p).idR$ but $\gamma'(q).idR \geq \gamma'(p).idR = \gamma(p).idR$. This contradicts Remark 4. □

Lemma 15. *There is no (fake) idR smaller than ℓ in a clean configuration.*

Proof. Let γ be a clean configuration. Assume there exists a process of idR smaller than ℓ. Let p be such a process such that $p.idR$ is minimum among all the processes and $p.level$ is minimum among all the processes having idR minimum.

Note that $p.idR \neq p$ and consequently $SelfRootOk(p)$ is false in γ. Hence (Remark 3), $KinshipOk(p, p.par)$ holds in γ. Since we take p of minimum idR, $p.idR \leq p.par.idR$ in γ. As $GoodIdR(p, p.par)$ implies that $p.idR \geq p.par.idR$, $p.idR = p.par.idR$. Now, $GoodLevel(p, p.par)$ implies that $p.level = p.par.level + 1$, which contradicts the minimality of $p.level$. □

For any process p, p can only set $p.idR$ to its own ID or copy the value of $q.idR$, where q is one of its neighbors. So, we have the following remark:

Remark 5. No fake ID is created during any step.

Lemma 16. *In a clean configuration, if the idR of a process p is ℓ, p is disable forever.*

Proof. Let γ be a clean configuration. Let p be a process with $\gamma(p).idR = \ell$. By Remark 4, only $J\text{-}action$ can be enabled in γ and its guard reduces to $\exists q \in \mathcal{N}_p, (q.idR < p.idR)$. But Lemma 15 ensures that this cannot be true, hence p is disabled in γ. Then, by Lemma 13 and Remark 5, this will be true forever. □

Corollary 4. *A clean configuration where $\forall p \in V, p.idR = \ell$, is terminal.*

Theorem 4. In a clean configuration, the system reaches a terminal configuration where $\forall p \in V, p.idR = \ell$ in at most $\mathcal{D}$ rounds.

Proof. Consider any execution e that starts from a clean configuration. In the following, we denote by ρ_i the first configuration of the ith round in e. We show by induction on the distance $d \geq 0$ between the processes and ℓ that $\forall p \in V$ such that $\|p, \ell\| \leq d$, $\rho_d(p).idR = \ell$.

Base case: If $\|p, \ell\| = 0$, $p = \ell$. Note that $KinshipOk(p, p.par)$ cannot hold in ρ_0 since $GoodIdR(p, p.par)$ would implies that $p.idR < p$ which is false by Lemma 15. Hence, from Remark 3, $SelfRoot(p) \wedge SelfRootOk(p)$ holds in ρ_0 and $\rho_0(p).idR = p = \ell$.

Induction step: Assume the property holds at some $d \geq 0$. If $\|p, \ell\| = d + 1$, $\exists q \in \mathcal{N}_p$ such that $\|q, \ell\| = d$. By induction hypothesis and by Lemma 16, $q.idR = \ell$ and q is disabled forever since ρ_d. If $p.idR = \ell$ in ρ_d, it remains so forever (Lemma 16). If $p.idR \neq \ell$ in ρ_d then $q.idR < p.idR$ (Lemma 15). Then, $J\text{-}action$ is enabled at p in ρ_d and remains enabled until p executes it (Lemma 14). As there is no fake ID smaller than ℓ (Lemma 15), $p.idR = \ell$ after p executes $J\text{-}action$, *i.e.*, after at most one round. Hence, $p.idR = \ell$ in ρ_{d+1}.

As $\mathcal{D} \geq \max\{\|p, \ell\|, p \in V\}$, in at most $\mathcal{D}$ rounds, the system reaches a configuration where $\forall p \in V, p.idR = \ell$. By Corollary 4, this configuration is terminal. □

Dirty Configurations.

In the previous section, we showed that, if the initial configuration is clean, the system reaches a terminal configuration in at most $\mathcal{D}$ rounds. But what happens if the initial configuration is dirty, *i.e.*, if there is a process p such that $EBroadcast(p)$ holds or $p.status \neq C$. In this section, we prove that starting from a dirty configuration, the system reaches a clean configuration in at most $3n$ rounds. More precisely, we show that a dirty configuration contains abnormal trees that are "cleaned" in at most $3n$ rounds. The system will be in a clean configuration afterwards.

Lemma 17. *In an dirty configuration, there exists at least one abnormal root.*

Proof. Let γ be a dirty configuration. Then, $\exists p \in V$ such that $p.status \neq C \vee EBroadcast(p)$. We search for an abnormal root.

1. If $p.status \in \{EB, EF\}$, using Observation 1, there is $q \in KPath(p)$ such that $q.status \in \{EB, EF\} \wedge Root(q)$. Then, $AbRoot(q) \vee SelfRoot(q)$. Now, $SelfRoot(q) \wedge q.status \in \{EB, EF\}$ implies $AbRoot(q)$. Hence, in all cases, $AbRoot(q)$ holds.
2. If $EBroadcast(p)$ holds, Lemma 8 applies and we are done. □

We have just shown that there are abnormal roots (and so abnormal trees) in dirty configurations. Below, we prove that these abnormal trees will disappear after three waves of "cleaning". After the first wave, an abnormal tree becomes dead (Theorem 5), after the second wave any abnormal root gets the status EF (Theorem 6) and finally after the third wave there is no more abnormal trees (Theorem 7), hence the system is in a clean configuration.

The following technical lemma is used in the proof of Theorem 5.

Lemma 18. *When EB-action is enabled at a process p, it remains enabled until p executes EB-action.*

Proof. Assume that EB-$action$ is enabled at a process p in a configuration γ, but p did not execute EB-$action$ during the step $\gamma \mapsto \gamma'$. Notice that p does not execute any action during this step, as guards are mutually exclusive. As EB-$action$ is enabled in γ, $\gamma(p).status = C$ and then, $\gamma'(p).status = C$.

First, assume $AbRoot(p)$ holds in γ. If $SelfRoot(p) \wedge \neg SelfRootOk(p)$ holds in γ and, as these predicates only depends on the local state of p and as p does not execute any action during the step, it also holds in γ': the action is still enabled in γ'. Otherwise, $\neg SelfRoot(p) \wedge \neg KinshipOk(p, p.par)$ holds in γ. These predicates only depends on the local state of p and its parent. Now, $Allowed(p.par)$ does not hold in γ because of p, so $p.par$ cannot execute R-$action$ nor J-$action$ during $\gamma \mapsto \gamma'$. Then, either $p.par$ executes EF-$action$, changes its status to EF and $GoodStatus(p, p.par)$ is false in γ', or it executes EB-$action$ and changes its status to EB. In these two cases, $EBroadcast(p)$ holds in γ'.

Now, assume $p.par.status = EB$, $p.par$ can only execute EF-$action$ and change its status to EF. Then, $GoodStatus(p, p.par)$ is false in γ', which implies that $EBroadcast(p)$ holds in γ'. □

Theorem 5. In at most n rounds, the system reaches a configuration where every abnormal tree (if any) is dead.

Proof. Consider any execution $e = \gamma_0, \ldots$. $\forall i > 0$, we denote by γ_{R_i} the last configuration of the ith round and so the first configuration of the $i + 1$th round of e. Moreover, let $\gamma_{R_0} = \gamma_0$ be the initial configuration. We show by induction on the length of the $KPath$s that, $\forall i \geq R_d$ ($d \geq 1$), $\forall p \in V$, if p is in an abnormal tree and $|KPath(p)| \leq d$ in γ_i, then p is dead in γ_i.

Base Case: If p is in an abnormal tree and $|KPath(p)| = 1$, p is an abnormal root. As no alive abnormal root is created (Lemma 2), if p is alive, it is an alive abnormal root since γ_{R_0} and if predicate $(p.status = C \wedge AbRoot(p))$ becomes false in some configuration, then it remains false forever. Hence, it is sufficient to show that any alive abnormal root is no more an alive abnormal root after one round (that is, from γ_{R_1}).

By definition, EB-$action$ is enabled at p in γ_{R_0} and p executes EB-$action$ during the first round (Lemma 18). Hence, p is dead at the end of the first round, and we are done.

Induction Hypothesis: Let $d \geq 1$. Assume that $\forall i \geq R_d$, $\forall p \in V$, if p is in an abnormal tree and $|KPath(p)| \leq d$ in γ_i, then p is dead in γ_i.

Induction Step: We first show that for every $p \in V$, for every $i \geq R_d$, if $(p.status = C \wedge |KPath(p)| \leq d + 1)$ is false in configuration γ_i, then for every $j \geq i$, $(p.status = C \wedge |KPath(p)| \leq d + 1)$ is false in configuration γ_j.

Assume by contradiction that the predicate "$p.status = C \wedge |KPath(p)| \leq d + 1$" is false in γ_j, but true in γ_{j+1} ($j \geq i$). By induction hypothesis, $|KPath(p)| = d + 1 > 1$ in γ_{j+1} (indeed, p is alive in γ_{j+1}). So, $\gamma_{j+1}(p).par \neq p$. So, let $q \in \mathcal{N}_p$ such that $\gamma_{j+1}(p).par = q$. By definition, $|KPath(q)| = d$ in γ_{j+1}. By induction hypothesis, $\gamma_{j+1}(q).status \in \{EB, EF\}$. Now, $p.status = C$ and $|KPath(p)| > 1$ in γ_{j+1}, so p is not an abnormal root in γ_{j+1}. Hence, $\gamma_{j+1}(q).status = EB$ (by Observation 1) and, consequently, $\gamma_j(q).status \in \{C, EB\}$.

- If $\gamma_j(q).status = EB$, then p does not execute any action in the step $\gamma_j \mapsto \gamma_{j+1}$ (otherwise, $\gamma_{j+1}(p).status \neq C$ or $\gamma_{j+1}(p).par \neq q$). Hence, $\gamma_j(p).status = \gamma_{j+1}(p).status = C$. By hypothesis, "$p.status = C \wedge |KPath(p)| \leq d+1$" is false in γ_j, so we have $|KPath(p)| > d+1$ in γ_j. Now, $\gamma_j(p).status = C$ and $\gamma_j(q).status = EB$, so $S\text{-}Trace(KPath(p)) = EB^+C$ in γ_j (Observation 1) and p is the only process in its $KPath$ that can execute an action in $\gamma_j \mapsto \gamma_{j+1}$. Hence, for every q such that $q \in KPath(p)$ in γ_j, we have $q \in KPath(p)$ in γ_{j+1}, and consequently $|KPath(p)| > d+1$ in γ_{j+1}. So $p.status = C \wedge |KPath(p)| \leq d+1$ is false in γ_{j+1}, a contradiction.
- If $\gamma_j(q).status = C$, then q is in an alive abnormal tree in γ_j (indeed, q executes $EB\text{-}action$ in $\gamma_j \mapsto \gamma_{j+1}$, and so Lemma 8 applies). As q is alive in γ_j, we have $|KPath(q)| > d$ in γ_j by induction hypothesis. Moreover, q is not an abnormal root (there is no more alive abnormal root after the first round, see the base case). Hence, the status of its parent in γ_j is EB. Now, $|KPath(q)| > d$ and $S\text{-}Trace(KPath(q)) = EB^+C$ in γ_j (Observation 1). So, q is the only one in its $KPath$ that executes an action in $\gamma_j \mapsto \gamma_{j+1}$ and this action is $EB\text{-}action$, which maintains the $KinshipOk$ relation. Hence, $|KPath(q)| > d$ in γ_{j+1} and consequently, $|KPath(p)| > d+1$ in γ_{j+1}, a contradiction.

Hence, for every process p, if $(p.status = C \wedge |KPath(p)| \leq d+1)$ is false in some configuration γ_i with $i \geq R_d$, then $(p.status = C \wedge |KPath(p)| \leq d+1)$ remains false forever.

Now, $EB\text{-}action$ is continuously enabled $\forall p$ such that p is alive $|KPath(p)| = d+1$ in γ_{R_d} (by induction hypothesis and Lemma 18). So, p becomes dead during the round and, $\forall j \geq R_{d+1}$, γ_j contains no alive process p such that $|KPath(p)| \leq d+1$.

$n \geq \max\{|KPath(p)|, \forall p \in V\}$. Hence, any process in an abnormal tree becomes dead in at most n rounds, and we are done. □

Lemma 19. *If $EF\text{-}action$ is enabled at a process p, it remains enabled until p executes $EF\text{-}action$.*

Proof. Assume by contradiction $EF\text{-}action$ is enabled at a process p in configuration γ and is not enabled in the next configuration γ', but p did not execute $EF\text{-}action$ during the step $\gamma \mapsto \gamma'$. Notice that p does not execute any action during this step, as guards are mutually exclusive. As $EFeedback(p)$ holds in γ, $\gamma(p).status = \gamma'(p).status = EB$. As $EFeedback(p)$ does not hold in γ' and no process can execute $J\text{-}action$ and choose a process of status EB as parent, $\exists q \in RealChildren_p$ such that $\gamma(q).status = EF$ and $\gamma'(q).status \neq EF$. Now, because $\gamma(q).status = EF$, q can only execute $R\text{-}action$. However, as $q \in RealChildren_p$, $KinshipOk(q,p)$ holds in γ and then q is not a root. So, q cannot execute any action and change its status during $\gamma \mapsto \gamma'$, a contradiction. □

Theorem 6. Let γ be a configuration containing abnormal trees and where all abnormal trees are dead. In at most n rounds from γ, the system reaches a configuration where the status of all abnormal roots is EF.

Proof. Consider any execution $e = \gamma_0, \ldots$ starting from a configuration that contains abnormal trees and where all abnormal trees are dead. $\forall i > 0$, we denote by γ_{R_i} the last configuration of the ith round and so the first configuration of the $i+1$th round. Moreover, let $\gamma_{R_0} = \gamma_0$ be the initial configuration.

Claim 1: $\forall p \in V, \forall i \geq R_0$, if $\gamma_i(p).status \neq EB$, then $\forall j \geq i, \gamma_j(p).status \neq EB$.

Assume by contradiction that $\gamma_j(p).status \neq EB$ and $\gamma_{j+1}(p).status = EB$, with $\gamma_j \mapsto \gamma_{j+1}$. Then, $p.status = C$ in γ_j and EB-$action$ is enabled at p in γ_j. So, p is in an alive abnormal tree in γ_j (Lemma 8), a contradiction to Lemma 3.

In any configuration γ, we denote by $MaxLengthKPath(p) = \max\{|KPath(q)|, q \in V \wedge p \in KPath(q)\}$. Again in γ, we define $L(p) = MaxLengthKPath(p) - |KPath(p)|$ and $EBL(p,k) \equiv p.status = EB \wedge L(p) = k$.

Claim 2: $\forall i \geq R_0$, if $EBL(p,k_i)$ holds in γ_i, then $\forall j \geq i, \forall k_j < k_i, \neg EBL(p,k_j)$ holds in γ_j. If $j = i$, $EBL(p,k_j)$ is false for $k_j < k_i$ because $L(p)$ cannot have two different values in a same configuration. Assume now $j > i$. The case $k_i = 0$ is direct. Assume $k_i > 0$. Assume by contradiction that $EBL(p,k_i)$ holds in γ_i and $EBL(p,k_j)$ holds in γ_j with $j > i$ and $k_j < k_i$. So, $\gamma_i(p).status = \gamma_j(p).status = EB$ and there are two cases:

- $p.status = EB$ in all the configurations between γ_i and γ_j. Consider the step $\gamma_i \mapsto \gamma_{i+1}$. Let q be any process such that $p \in KPath(q)$ in γ_i. So, $KPath(q) = q_0 \ldots q_i = p \ldots q_k = q$ and S-$Trace(KPath(q)) = EB^+EF^*$ in γ_i. There is a unique process in $KPath(q)$ that can execute an action in $\gamma_i \mapsto \gamma_{i+1}$ (the only one of status EB with children of status EF). If it executes an action, it is EF-$action$ which maintains $KinshipOk$ relation. Hence, $\forall q' \in KPath(q)$ in γ_i, $q' \in KPath(q)$ in γ_{i+1}. We can apply this latter property to every process r such that $p \in KPath(r)$ and $|KPath(r)| = MaxLengthKPath(p)$ in γ_i: $p \in KPath(r)$ in γ_{i+1} and the value of $|KPath(r)|$ in γ_{i+1} is greater than or equal to the value of $|KPath(r)|$ in γ_i. So, $EBL(p,k_{i+1})$ holds with $k_{i+1} \geq k_i$. Applying the same argument on step $\gamma_{i+1} \mapsto \gamma_{i+2}$, *etc.*, until step $\gamma_{j-1} \mapsto \gamma_j$, we obtain that $EBL(p,k_j)$ is true in γ_j with $k_j \geq k_i$, a contradiction.
- There is a configuration between γ_i and γ_j where $p.status \neq EB$. So, $\exists x$ such that $i < x < j$, $\gamma_x(p).status \neq EB$ and $\gamma_{x+1}(p).status = EB$. This contradicts Claim 1.

We show by induction that $\forall i \geq R_d$ with $d \geq 1, \forall p \in V, \forall k \leq d-1$, $EBL(p,k)$ is false in γ_i.

Base case: There are three cases:

1. If $L(p) = 0$ in γ_{R_0} and $\gamma_{R_0}(p).status = EB$, then EF-$action$ is enabled at p in γ_{R_0}, p executes EF-$action$ during the first round, by Lemma 19 and p gets status EF. By Claim 1, $p.status$ remains different from EB forever and $EBL(p,0)$ is false in $\gamma_i, \forall i \geq R_1$.
2. If $\gamma_{R_0}(p).status \neq EB$, $p.status \neq EB$ forever (Claim 1) and then $EBL(p,0)$ is false forever.
3. If $EBL(p,k)$ holds in γ_{R_0} with $k > 0$, $EBL(p,0)$ is false forever (Claim 2).

Induction hypothesis: $\forall i \geq R_d$ with $d \geq 1, \forall p \in V, \forall k \leq d-1$, $EBL(p,k)$ is false in γ_i.

Induction step: There are four cases:

1. If $L(p) = d$ and $\gamma_{R_d}(p).status = EB$, $\forall q \in RealChildren_p$ in γ_{R_d}, $L(q) < d$ by definition and $\gamma_{R_d(q)}.status \neq EB$ by induction hypothesis. Now, the trees are dead, so $\gamma_{R_d}(q).status = EF$. Hence, EF-$action$ is enabled at p in γ_{R_d}, p executes EF-$action$ during the round (Lemma 19) and gets status EF. By Claim 1, $p.status \neq EB$ forever so $EBL(p,d)$ is false at the end of the $d+1$th round and remains false forever.
2. If $L(p) = d$ and $\gamma_{R_d}(p).status \neq EB$, then $p.status \neq EB$ forever (Claim 1). So, $EBL(p,d)$ is false forever.

3. If $L(p) < d$, by induction hypothesis $\gamma_{R_d}(p).status \neq EB$ and we conclude as in case 2.
4. If $EBL(p, k)$ holds in γ_{R_d} with $k > d$, $EBL(p, i)$ is false forever $\forall i \leq d$ (Claim 2).

With $d = n$, we have $\forall i \geq R_n$, $\forall p \in V$, $\forall k \leq n - 1$, $EBL(p, k)$ is false in γ_i: hence, in at most n rounds, there is no more process of status EB in abnormal trees, those ones being dead. So, all processes (and in particular the abnormal roots) in abnormal trees have status EF. □

Lemma 20. *If all abnormal trees are dead and R-$action$ is enabled at a process p, then R-$action$ remains enabled at p until p executes it.*

Proof. Let γ be a configuration, where all abnormal trees are dead. Assume, by contradiction, that R-$action$ is enabled at a process p in a configuration γ and is not enabled in the next configuration γ', but p did not execute R-$action$ during the step $\gamma \mapsto \gamma'$. Notice that p does not execute any action during this step, as guards are mutually exclusive.

As R-$action$ is enabled in γ and p does not execute an action during the step, $\gamma(p).status = \gamma'(p).status = EF$.

If $SelfRoot(p) \wedge \neg SelfRootOk(p)$ holds in γ, it also holds in γ' because p does not execute an action between γ and γ' and these predicates only depends on the local state of p.

Otherwise $\neg SelfRoot(p) \wedge \neg KinshipOk(p, p.par)$ holds in γ. Let $q = p.par$. If q does not execute an action between γ and γ', p is still an abnormal root. Otherwise, three cases are possible:

- $\neg GoodIdR(p, q)$ holds in γ.
  1. First, if $\gamma(p).idR < \gamma(q).idR$. If q executes EB-$action$ or EF-$action$ during the step, the idR of q does not change, so $\gamma'(p).idR < \gamma'(q).idR$, and $AbRoot(p)$ holds in γ'. Otherwise q executes R-$action$ or J-$action$. Then $\gamma'(q).status = C$, so $\neg GoodStatus(p, q)$ and $AbRoot(p)$ holds in γ'.
  2. If $\gamma(p).idR \geq p$, the idR is not modified during the step, so $\gamma'(p).idR = \gamma(p).idR \geq p$ and $AbRoot(p)$ holds in γ'.
- $\neg GoodLevel(p, q)$ holds in γ. Then $\gamma(p).idR = \gamma(q).idR$ but $\gamma(p).level \neq \gamma(q).level + 1$. If q executes EB-$action$ or EF-$action$, its idR and its $level$ do not change, so $\gamma'(p).idR = \gamma'(q).idR$ and $\gamma'(p).level \neq \gamma'(q).level + 1$, so $AbRoot(p)$ holds in γ'. Otherwise, q executes R-$action$ or J-$action$. Then $\gamma'(q).status = C$, so $\neg GoodStatus(p, q)$ and $AbRoot(p)$ holds in γ'.
- $\neg GoodStatus(p, q)$ holds in γ. Then $\gamma(q).status = C$, and q can only execute EB-$action$ or J-$action$ between γ and γ'. If q executes EB-$action$ then $EBroadcast(q)$ holds in γ, so q is in an abnormal tree (Lemma 8). But, by hypothesis, all abnormal trees are dead in γ, so $\gamma(q).status \neq C$, a contradiction. If q executes J-$action$ then $\gamma'(q).status = C$, so $\neg GoodStatus(p, q)$ and $AbRoot(p)$ holds in γ'.

Thus, $\gamma'(p).status = EF$ and $AbRoot(p)$ holds in γ' and, consequently, $Allowed(p)$ is false in γ'. So $\exists q \in \mathcal{N}_p$ such that $q \in Children_p \wedge \neg KinshipOk(q, p)$ holds in γ' but $\gamma'(q).status = C$. Two cases are possible:

- If $q \notin Children_p$ in γ, then q executes J-$action$ during the step $\gamma \mapsto \gamma'$ and $Min_q = p$. But $\gamma(p).status = EF$, a contradiction.
- Otherwise $q \in Children_p$ in γ and $\gamma(q).status \neq C$. q executes either EF-$action$ and $\gamma'(q).status = EF$, or R-$action$ and $\gamma'(q).par \neq p$, so $q \notin Children_p$ in γ', a contradiction. □

Definition 13 (Abnormal process)**.** A process p is called *abnormal* process if and only if p belongs to an abnormal tree. p is said to be *normal*, otherwise.

As no process can join a dead abnormal tree (Remark 1) and no alive abnormal tree can be created (Lemma 3), we have the following remark:

Remark 6. In a configuration where every abnormal tree is dead, the number of abnormal processes can only decrease.

Theorem 7. Starting from a configuration where every abnormal tree is dead and the status of their roots is EF, there is no more abnormal processes in at most n rounds.

Proof. Let γ_0 be a configuration where all abnormal trees are dead and the status of their roots is EF. By Observation 1, all abnormal processes have status EF in γ_0. So, from γ_0, no process can be ever an abnormal process with a status different of EF (such a process can only execute $R\text{-}action$, then it is a normal process forever, by Lemma 3). Then, by definition, the number of abnormal processes in γ_0 is at most n. Moreover, by Remark 6, it is sufficient to show that in any configuration γ_k reachable from γ_0, if the number of abnormal processes is not null, then at least one of them becomes normal within the next round.

So, let assume that some process p is abnormal in γ_k. Then, $\gamma_k(p).status = EF$. By Observation 1 and Lemma 20, the initial extremity r of $KPath(p)$ is an abnormal process (of status EF) and executes $R\text{-}action$ within the next round. After executing $R\text{-}action$, r is normal (actually, r becomes a self root), and we are done. □

By definition, the root of a normal tree has the status C. So, by Observation 1, we have:

Remark 7. Every process has the status C in a configuration containing no abnormal processes. Moreover, this configuration is clean.

Using Lemma 17 and Theorems 5 to 7, we can conclude:

Theorem 8. In at most $3n$ rounds, the system reaches a clean configuration.

Then, using Theorems 4 and 8 we get:

Theorem 9 (Round Complexity)**.** In at most $3n + \mathcal{D}$ rounds, the system reaches a terminal configuration.

4.3.2 Worst Case Analysis of the Stabilization Time

Lower Bound on the Worst Case Stabilization Time in Rounds.

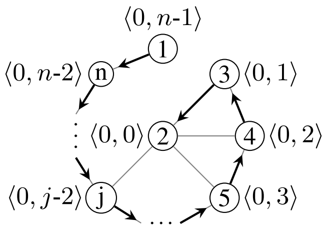
We now show that the bound proposed in Theorem 9 cannot be improved. To see this, we exhibit a construction that gives, $\forall n \geq 4$, $\forall \mathcal{D}, 2 \leq \mathcal{D} \leq n-2$, a network of n processes whose diameter is $\mathcal{D}$, from which there is a possible synchronous execution that lasts exactly $3n+\mathcal{D}$ rounds. (Recall that every synchronous execution is possible under the distributed unfair daemon.)

We consider a network $G = (V, E)$ composed of n processes $V = \{p_1, \ldots, p_n\}$ such that p_i has ID i, $\forall i \in [1..n]$. Figure 8a shows the system in its initial configuration. In details, processes $p_1, p_n, \ldots, p_2$ form a chain, *i.e.*, $\{p_1, p_n\} \in E$ and $\{p_i, p_{i-1}\} \in E \; \forall i = 3 \ldots n$.

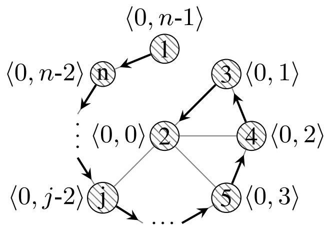
We add k "*legs*", with $2 \leq k \leq n-2$, as follows:

If $k = n-2$**, then** $\{p_2, p_1\} \in E$ and $\forall i \in [4..n]$, $\{p_2, p_i\} \in E$,

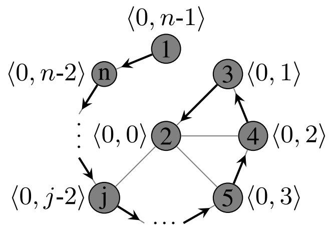
Otherwise $\forall i \in [4..k+3]$, $\{p_2, p_i\} \in E$.



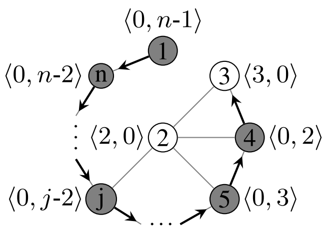
(a) The initial configuration. $\{p_2, p_j\}$ is the “last” leg ($j = k+3$).



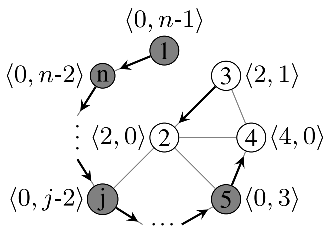
(b) In n rounds, the EB-wave reaches p_1.



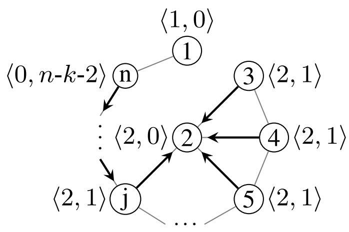
(c) In n rounds, p_2 gets status EF.



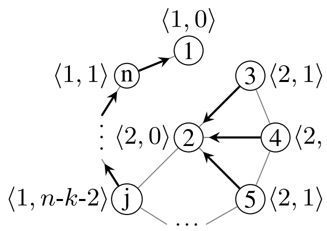
(d) p_2 and p_3 sequentially execute $R\text{-}action$.



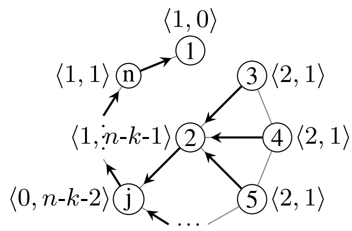
(e) p_3 executes $J\text{-}action$ and p_4 simultaneously executes $R\text{-}action$.



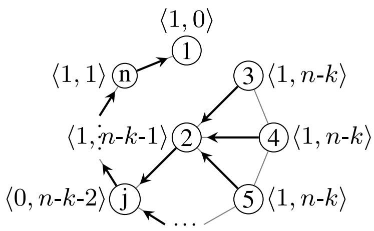
(f) In $n-3$ rounds, the cleaning is finished.



(g) In $n-k-2$ rounds, processes n to $k-3$ joins $Tree(1)$.



(h) Processes p_2 and p_{k-2} simultaneously execute $J\text{-}action$.



(i) In one round, the system reaches a terminal configuration where p_1 is the leader.

Figure 8: An example in $3n + \mathcal{D}$ rounds

Notice that the diameter of the network is $n-k$ and can be adjusted by adding or removing some legs.

We assume the following initial configuration:

- $p_i.idR = 0 \ \forall i \in [1..n]$,
- $p_1.level = n-1$ and $p_1.par = p_n$,
- $p_2.par = p_2$ and $p_2.level = 0$,
- $p_i.level = i-2$ and $p_i.par = p_i - 1, \forall i \in [3..n]$.

We consider a synchronous daemon, *i.e.*, in a configuration γ, every process in $Enabled(\gamma)$ is selected by the daemon to execute an action. So, in this case, every round lasts exactly one step.

The execution is then as follows:

- $p_2, p_3, p_4 \dots p_n, p_1$ sequentially execute $EB\text{-}action$: n rounds. (See Figure 8b.)
- $p_1, p_n, p_{n-1}, \dots, p_2$ sequentially execute $EF\text{-}action$: n rounds. (See Figure 8c.)
- p_2 and p_3 sequentially execute $R\text{-}action$: 2 rounds. (See Figure 8d.)
- For $i = 4 \dots n$, simultaneously p_i and p_{i-1} respectively executes $R\text{-}action$ and $J\text{-}action$, in particular, p_{i-1} joins $Tree(p_2)$: $n-3$ rounds. (See Figures 8e and 8f.)

– p_1 executes R-*action* and p_n executes J-*action* simultaneously: 1 round.
– For $i = n \dots k{+}3$, i executes J-*action* to join $Tree(1)$: $n{-}k{-}2$ rounds. (See Figure 8g.)
– p_2 and p_{k+2} simultaneously execute J-*action* to join $Tree(1)$: 1 round. (See Figure 8h.)
– $p_3, \dots, p_{k+1}$ simultaneously execute J-*action* and then the configuration is terminal: 1 round. (See Figure 8i.)

Hence, the execution lasts exactly $3n + (n - k) = 3n + \mathcal{D}$ rounds. Using Theorem 9 we can conclude:

Theorem 10. In the worst case, the round complexity of $\mathcal{LE}$ is exactly $3n + \mathcal{D}$ rounds.

Lower Bound on the Worst Case Stabilization Time in Steps.

We show that the bound given in Theorem 1 can be asymptotically matched, *i.e.*, we give an example of possible execution that stabilizes in $\Omega(n^3)$ steps, for every $n \geq 4$.

We consider a network $G = (V, E)$ composed of n processes $V = \{p_1, \dots, p_n\}$ such that p_i has ID $n + i$, $\forall i \in [1..n]$. Figure 9a shows the network in this initial configuration. In details, there are $2n - 3$ edges: $\{p_i, p_{i+1}\}$ $\forall i = 1 \dots n - 2$ and $\{p_i, p_n\}$ $\forall i = 1 \dots n - 1$. (Notice that the diameter of this network is 2.) The initial configuration is as follows:
– $p_i.idR = i$ $\forall i \in [1..n - 1]$, and $p_n.idR = 2n$.
– $p_i.par = p_i$, $p_i.level = 0$ and $p_i.status = C$ $\forall i \in [1..n]$.

We consider the following execution:
– For $i = n - 1 \dots 1$, $(i - -)$, we clean $Tree(p_i)$ the following way:

1. For $j = n - 2 \dots i$, $(j - -)$, [1]
   a) For $k = j + 1 \dots n - 1$, $(k + +)$, [2]
      – p_k joins $Tree(p_j)$.

   This part lasts $\sum_{k=1}^{n-1-i} k$ steps.
2. $p_i, p_{i+1}, \dots, p_{n-1}$ sequentially execute EB-*action*: $n - i$ steps.
3. $p_{n-1}, p_{n-2}, \dots, p_i$ sequentially execute EF-*action*: $n - i$ steps.
4. $p_i, p_{i+1}, \dots, p_{n-1}$ sequentially execute R-*action*: $n - i$ steps.

Figures 9e to 9h show the cleaning of $Tree(p_{n-3})$.

– After all abnormal trees have been cleaned, processes p_{n-1} to p_2 join $Tree(p_1)$ similarly as Case 1: $\sum_{i=1}^{n-2} i$ steps (Figure 9j).
– p_n executes J-*action* to join $Tree(p_1)$: 1 step (Figure 9k).

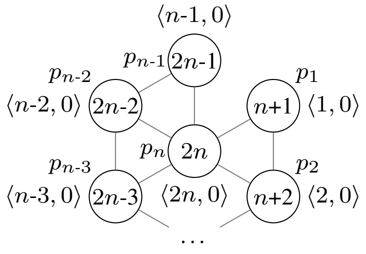
Hence, the complete execution lasts:

$$\left(\sum_{i=1}^{n-1}(3(n-i) + \sum_{k=1}^{i-1} k)\right) + \left(\sum_{i=1}^{n-2} i\right) + 1 = \frac{n^3}{6} + \frac{5}{2}n^2 - \frac{11}{3}n + 2 \text{ steps}$$

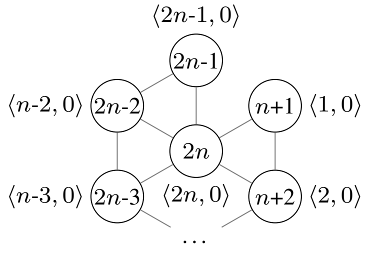
So, there exists an execution in $\Omega(n^3)$. Using Theorem 3, we can conclude:

Theorem 11. In the worst case, the step complexity of $\mathcal{LE}$ is in $\Theta(n^3)$ steps.

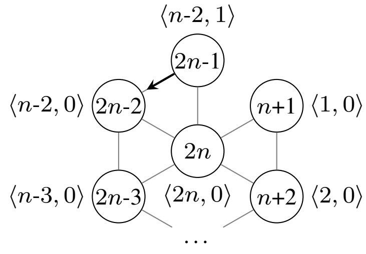
1. Of course, when $n - 2 < i$, there is no iteration.
2. Of course, when $j + 1 > n - 1$, there is no iteration.



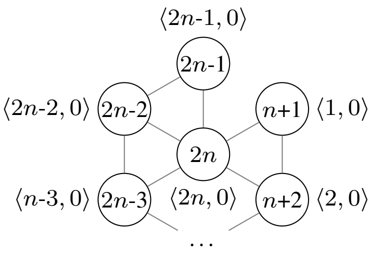
(a) The initial configuration



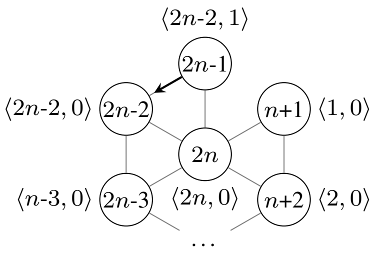
(b) In three steps, p_{n-1} becomes normal



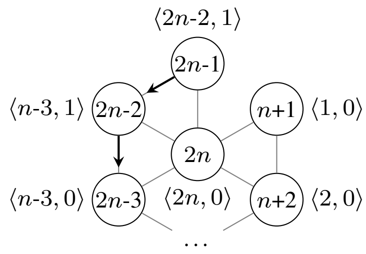
(c) p_{n-1} executes $J\text{-}action$ and joins $Tree(p_{n-2})$



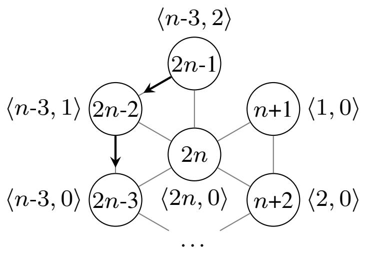
(d) In six steps, the abnormal tree rooted in p_{n-2} is cleaned



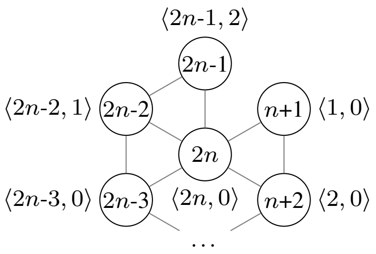
(e) p_{n-1} executes $J\text{-}action$ and joins the normal tree $Tree(p_{n-2})$



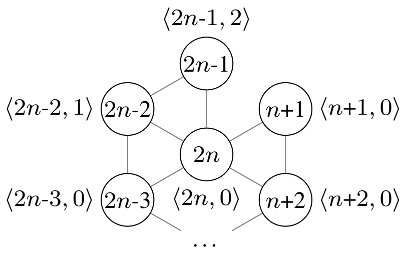
(f) p_{n-2} executes $J\text{-}action$ and joins the abnormal tree $Tree(p_{n-3})$



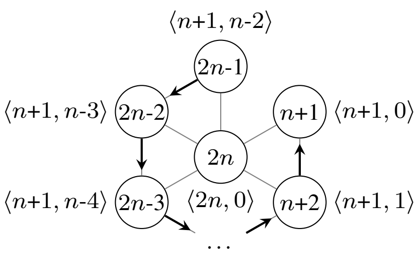
(g) p_{n-1} executes $J\text{-}action$ and updates its idR to $n-3$.



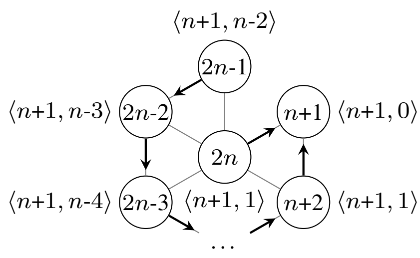
(h) In nine steps, the abnormal tree rooted in p_{n-3} is cleaned



(i) There is no more abnormal trees



(j) In $\sum_{i=1}^{n-2} i$ steps, processes p_{n-1} to p_2 elect p_1



(k) In one step, the system reaches a terminal configuration where p_1 is leader.

Figure 9: An example in $\Omega(n^3)$ steps

— 5 —

Step Complexity of Algorithm $\mathcal{DLV}$

In this chapter, we study the step complexity of the algorithm given in [9], called here $\mathcal{DLV}$.[1] Below, we show that its stabilization time is not polynomial in steps.

First, we give the code of algorithm $\mathcal{DLV}$ and an informal explanation of its main principles in Section 5.1. Then, in Section 5.2 we give an example of a class of network in which there is a possible execution that stabilizes in $\Omega(n^4)$ steps. Finally, in Section 5.3, we generalize the previous example to a class of network where there is a possible execution that stabilizes in $\Omega(n^{\alpha+1})$ for any $\alpha \geq 3$.

5.1 Overview of $\mathcal{DLV}$

First, Algorithm $\mathcal{DLV}$ uses priorities. Each of its actions is given with priority number. When an enabled process is selected by the daemon, it only executes its enabled action with the lowest priority number.

Algorithm $\mathcal{DLV}$ (refer to Algorithm 2) elects the process of minimum ID, ℓ, and builds a breadth-first spanning tree rooted at ℓ. To ensure that every process knows which one is elected, it maintains a variable $leader$ to save its current leader. Variables $parent$ and $level$ are used to represent the tree. The key of a process p is the combination of its two variables $p.leader$ and $p.level$. Notice that the keys are ordered by a lexical order.

When a process p has a neighbor with a smaller key, p executes action J, gets the successor key of the smaller such neighbor ($BestNbrKey(p)$) and chooses this latter as parent. Notice that, contrary to our algorithm, p can execute action J and change its parent if there is a process with the same leader but with a level smaller than $p.level - 1$, in order to build a breadth-first spanning tree.

As in $\mathcal{LE}$, they define a "good relation" between a process p and its parent called $IsTrueChld(p)$. It ensures that the key of p is the successor key of its parent and that its leader is smaller than its own ID. Then, a maximal set of processes linked by $parent$ pointers and satisfying $IsTrueChld$ relation defines a tree. The root of this tree can be a *true root* ($IsTrueRoot(p)$), *i.e.*, the key of p is its self key ($\langle p, 0\rangle$). In this case, they said that it is a *normal tree*. Otherwise, the root is a *false root* ($IsFalseRoot(p)$), *i.e.*, neither a true root nor a true child, and they said that it is an *abnormal tree*.

Color waves.

The main difference between $\mathcal{DLV}$ and $\mathcal{LE}$ is the way to deal with these abnormal trees. Instead of using a status and a three waves cleaning, $\mathcal{DLV}$ uses color waves. More precisely, each process has a variable $color$, either 1 or 2. A process can only change its parent to a neighbor of color 2 and after executing action J, the process gets color 1.

A process p of color 2 cannot change its color to 1 when it has possible recruits ($Recruits(p) \neq \emptyset$), *i.e.* there are some neighbors with a bigger key that may choose p as parent later. Furthermore, a process can change its color, executing actions $C1$ or $C2$, if it has the same color than its parent (it is trivially satisfied for every true root) and if all of its true children have the other color.

1. $\mathcal{DLV}$ stands for "Datta, Larmore and Vemula."

Algorithm 2 Algorithm $\mathcal{DLV}$ [9] for every process p

Variables

$p.leader \in \mathbb{N}$, $p.level \in \mathbb{N}$, $p.key = \langle p.leader, p.level\rangle$, $p.parent \in \mathcal{N}_p \cup \{p\}$
$p.color \in \{1, 2\}$, $p.done \in \mathbb{B}$

Macros

$SelfKey(p)$	$\equiv$	$\langle p, 0\rangle$
$SuccKey(p)$	$\equiv$	$\langle p.leader, p.level + 1\rangle$
$BestNbrKey(p)$	$\equiv$	$\min\{q.key \mid (q \in \mathcal{N}_p) \wedge (SuccKey(q) < SelfKey(p)) \wedge (q.color = 2)\}$
$TrueChldrn(p)$	$\equiv$	$\{q \in \mathcal{N}_p \mid (q.parent = p) \wedge (q.key = SuccKey(p))\}$
$FalseChldrn(p)$	$\equiv$	$\{q \in \mathcal{N}_p \mid (q.parent = p) \wedge (q.key \neq SuccKey(p))\}$
$Recruits(p)$	$\equiv$	$\{q \in \mathcal{N}_p \mid q.key > SuccKey(p)\}$

Predicates

$IsTrueRoot(p)$	$\equiv$	$p.key = SelfKey(p)$
$IsTrueChld(p)$	$\equiv$	$(p.key = SuccKey(p.parent) \wedge (p.leader < p)$
$IsFalseRoot(p)$	$\equiv$	$\neg IsTrueRoot(p) \wedge \neg IsTrueChld(p)$
$Done(p)$	$\equiv$	$(Recruits(p) = \emptyset) \wedge (\forall q \in TrueChldrn(p), q.done)$
$ColorFrozen(p)$	$\equiv$	$IsTrueRoot(p) \wedge p.done$

Guards

$Join(p, q)$	$\equiv$	$(IsFalseRoot(p) \vee (SuccKey(q) < p.key)) \wedge (q.color = 2)$ $\wedge (q.key = BestNbrKey(p)) \wedge (FalseChldrn(p) = \emptyset)$
$Reset(p)$	$\equiv$	$IsFalseRoot(p)$
$Color1(p)$	$\equiv$	$(p.color = 2) \wedge \neg ColorFrozen(p) \wedge (p.parent.color = 2)$ $\wedge (Recruits(p) = \emptyset) \wedge (\forall q \in TrueChldrn(p), q.color = 1)$
$Color2(p)$	$\equiv$	$(p.color = 1) \wedge \neg ColorFrozen(p) \wedge (p.parent.color = 1)$ $\wedge (\forall q \in TrueChldrn(p), q.color = 2)$
$UpdateDone(p)$	$\equiv$	$p.done \neq Done(p)$

Actions

J (priority 1)	::	$Join(p, q)$	$\rightarrow$	$p.key = SuccKey(q)$; $p.parent = q$; $p.color = 1$; $p.done =$ `false`;
R (priority 2)	::	$Reset(p)$	$\rightarrow$	$p.key = SelfKey(p)$; $p.parent = p$; $p.color = 2$; $p.done =$ `false`;
$C1$ (priority 3)	::	$Color1(p)$	$\rightarrow$	$p.color = 1$; $p.done = Done(p)$;
$C2$ (priority 3)	::	$Color2(p)$	$\rightarrow$	$p.color = 2$; $p.done = Done(p)$;
UD (priority 4)	::	$UpdateDone(p)$	$\rightarrow$	$p.done = Done(p)$;

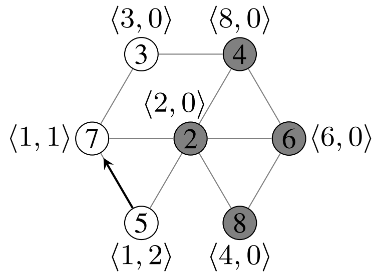
To add a new level in the tree, the leaves must change their color to 2. A first wave of actions $C1$ is initiated by the parents of the leaves and absorbed by the root. Then, a second wave of actions $C2$ is initiated by the leaves and also absorbed by the root. When the leaves have color 2, their neighbors can join the tree. Now, the priorities on actions prevent a false root to change its color and, so, to absorb a color wave. Moreover, every true root can always absorb a color wave.

Therefore, the colors of the processes in an abnormal tree eventually alternate, *i.e.*, the parents and their real children do not have the same color, and no more process can join the tree: the tree is *color locked*. Then, the root eventually resets to a true root executing action R.

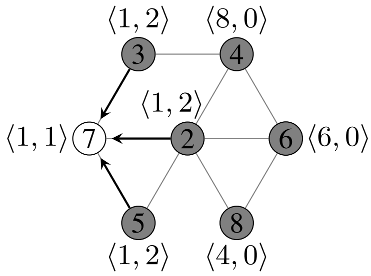
Once all abnormal trees have been removed, ℓ is a true root and regularly absorb color waves allowing then the leaves of its tree to recruit processes.

Figure 10 shows an example of execution with the cleaning of an abnormal tree.

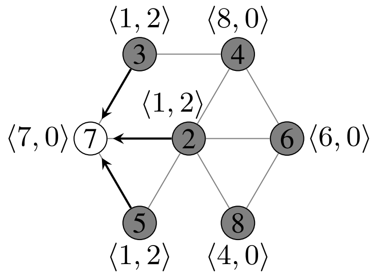
Finally, in $O(n)$ rounds, ℓ is elected and a breadth-first spanning tree rooted at ℓ is built. Notice that the color waves might never end. A mechanism ensure the silence of the algorithm using the Boolean variable $done$ and action UD. When a process p believes that the construction of the final tree is finished (because it can no more recruits other processes) and all its true children q (if any) have set their variables $q.done$ to `true`, $p.done$ is set to `true`. Moreover, a



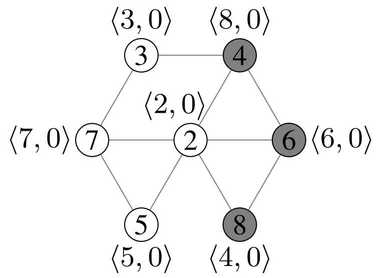
(a) Initial configuration. 1 is a fake ID.



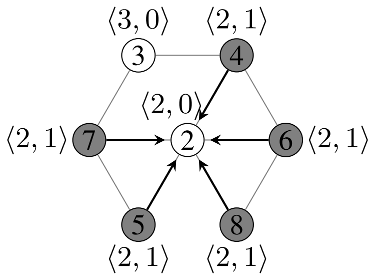
(b) 2 and 3 have executed action J and chosen 7 (of color 2) as parent. 5 has changed its color to 2 executing action $C2$.



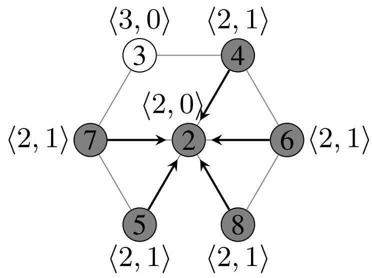
(c) The tree of 7 was color locked. Then, 7 executed action R.



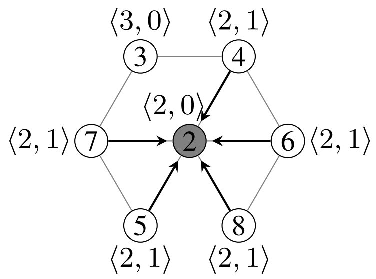
(d) 2, 3 and 5 were false roots and have executed action R.



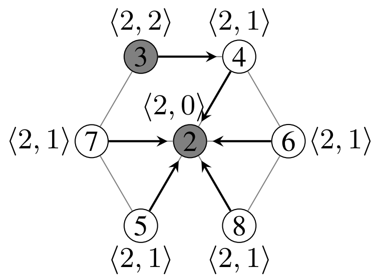
(e) 4, 5, 6, 7 and 8 have executed action J and chosen 2 as parent.



(f) 3 cannot join the tree of 2 because all its neighbors have color 1. 2 has changed its color to 1 by executing action $C1$.



(g) 4,5,6,7 and 8 have changed their color to 2 by executing action $C2$.



(h) Then, 3 was able to execute action J and join the tree of 2.

Figure 10: Example of execution of algorithm $\mathcal{DLV}$. The ID is represented inside the node. The label next to a node shows its key. The arrows represent $parent$ pointers. No arrow exits a node if its parent is itself. The filling represents the color: gray for 1 and white for 2.

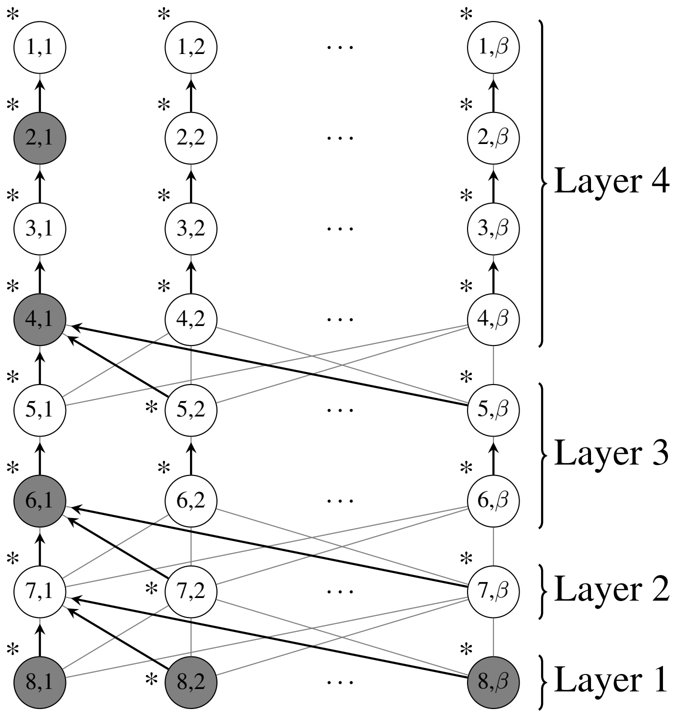


Figure 11: Initial configuration. The $leader$ of a process is 0 if it gets a star or its own ID otherwise. $level$ is not represented as it is always correct.

true root r cannot change its color if $r.done$ holds. We said that r is *color frozen*. Thus, after the completion of the final tree construction, the value `true` is propagated bottom-up in the tree into the $done$ variables, and in $O(\mathcal{D})$ rounds, the system reaches a terminal configuration.

5.2 Example in $\Omega(n^4)$ steps

We consider a network made of $n = L \times \beta$ processes with $L = 8$ and $\beta \geq 2$: $p_{(1,1)}, p_{(1,2)}, \ldots, p_{(1,\beta)}, p_{(2,1)}, p_{(2,2)}, \ldots, p_{(2,\beta)}, \ldots, p_{(8,1)}, p_{(8,2)}, \ldots, p_{(8,\beta)}$ such that the ID of $p_{(i,j)}$ is $(i-1)\beta + j, \forall i \in [1 \ldots 8], \forall j \in [1 \ldots \beta]$. Notice that 0 is a fake ID smaller than every ID in the network.

Figure 11 shows the structure of the network and the initial configuration. In details, the processes form β columns: $\forall i \in [2 \ldots 8], \forall j \in [1 \ldots \beta], \{p_{(i-1,j)}, p_{(i,j)}\} \in V$. Moreover, there are three complete bipartite subgraphs: $\forall j \in [1 \ldots \beta], \forall j' \in [1 \ldots \beta], j' \neq j$, $\{p_{(4,j)}, p_{(5,j')}\}$, $\{p_{(6,j)}, p_{(7,j')}\}$ and $\{p_{(7,j)}, p_{(8,j')}\}$ are in V. These bipartite subgraphs split the network in four layers:

- Layer 1: line 8
- Layer 2: line 7
- Layer 3: lines 5 and 6
- Layer 4: lines 1 to 4

We choose the following initial configuration.

- For $i \in [1 \ldots 8]$, $j \in [1 \ldots \beta]$, $p_{(i,j)}.leader = 0$, $p_{(i,j)}.level = i$ and $p_{(i,j)}.done = \texttt{false}$
- For $j \in [1 \ldots \beta]$,
  - $p_{(1,j)}.parent = p_{(1,j)}$
  - $p_{(5,j)}.parent = p_{(4,1)}$
  - $p_{(7,j)}.parent = p_{(6,1)}$
  - $p_{(8,j)}.parent = p_{(7,1)}$
  - For $i \in [2 \ldots 4] \cup \{6\}$, $p_{(i,j)}.parent = p_{(i-1,j)}$

- For $i \in [1 \ldots 8]$, $p_{(i,1)}.color = (i \mod 2) + 1$
- For $j \in [2 \ldots \beta]$,
  - $p_{(8,j)}.color = 1$
  - For $i \in [1 \ldots 7]$, $p_{(i,j)}.color = 2$

We consider an unfair daemon which selects the enabled processes according to function DAEMON given in Algorithm 3. In this algorithm , $top(i)$ (respectively $bottom(i)$) is the number of the first line (respectively last line) of layer i. More precisely:

$$top(i) = L - 2^{i-1} + 1$$

$$bottom(i) = \begin{cases} top(1) & \text{if } i = 1 \\ top(i-1) - 1 & \text{if } i > 1 \end{cases}$$

In BUILD$(layer, column)$, all the processes of lines $top(layer)$ to 8 execute line by line action J. Notice that the processes of line $top(layer)$ choose $p_{(top(layer)-1,column)}$ as parent. In RESET$(layer, column)$, processes $p_{(top(layer+1),column)}$ to $p_{(bottom(layer+1),column)}$ execute action R (except for layer 1 where all the processes of line 8 also execute action R). Then, RESET$(layer - 1, i)$ and BUILD$(layer - 1, i + 1)$ are called for each column $i = 1 \ldots \beta - 1$. Finally, RESET$(layer - 1, \beta)$ is executed.

The idea is to reset a branch of the tree and then, rebuild symmetrically the tree on the next column: a process chooses as parent the neighbor of smaller key, *i.e.*, the extreme left neighbor one line above having 0 as $leader$. More precisely, a first sequence of actions R resets the first column and the layer 1 (Figure 12). Then, the layer 1 is rebuilt on the second column (BUILD(1,2)) and reset again (Figure 13) and so forth until the last column. Then, the tree is rebuilt since the second layer on the second column (BUILD(2,2)) and the extreme left branch is reset (Figure 14) and so on.

To better understand the algorithm with its numerous recursive calls, a step by step execution of function DAEMON is provided in Appendix A. The reader can follow the execution on an empty figure given with the listing.

We count how many times processes $p_{(8,.)}$ executes action R:

- Each process $p_{(8,.)}$ executes once action R in RESET$(layer, column)$, when $layer = 1$ (line 15 of Algorithm 3): at least β processes execute action R.
- RESET$(3, column)$ is called β times by DAEMON.
- RESET$(2, column)$ is called β times by RESET$(3, column)$.
- RESET$(1, column)$ is called β times by RESET$(2, column)$.

Hence, action R is executed β^4 times by the processes of line 8. Now, $\beta = n/8$. Hence we can conclude:

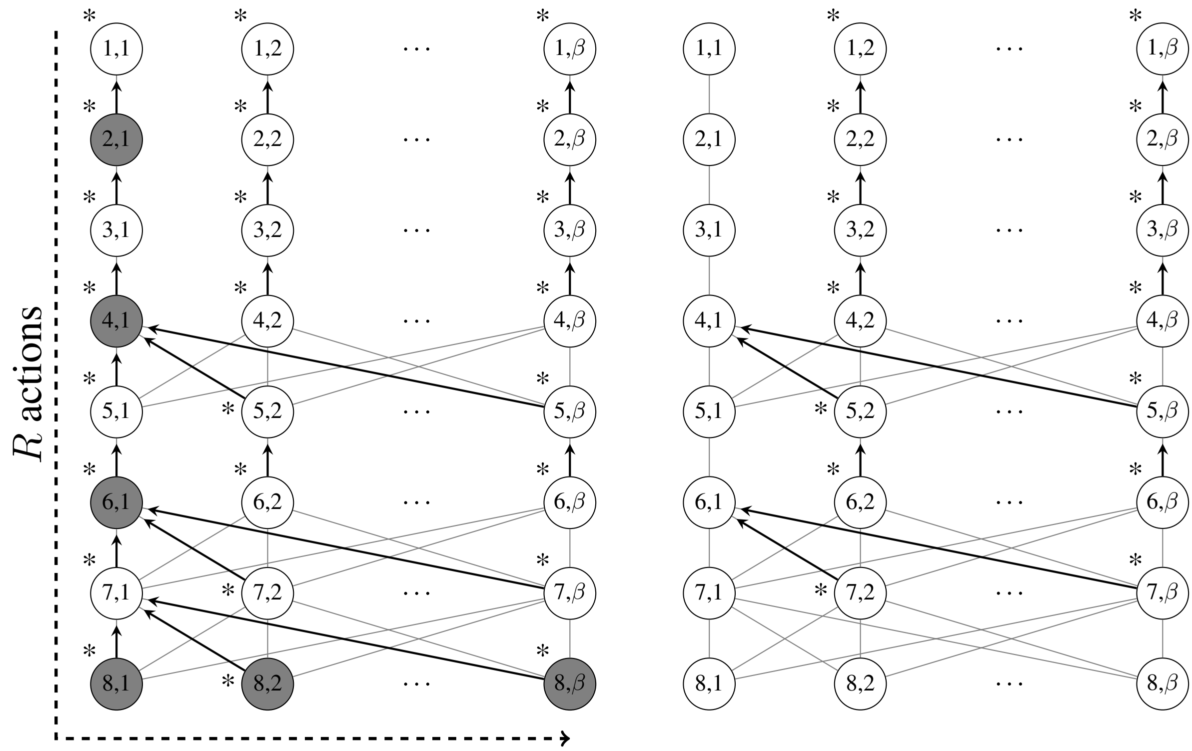
Theorem 12. For every $\beta \geq 2$, there exists a network of $n = 8 \times \beta$ processes in which there exists a possible execution that stabilizes in $\Omega(n^4)$ steps.

5.3 Generalization to an example in $\Omega(n^{\alpha+1})$ steps

Starting from E_α ($\alpha \geq 4$), an example in $\Omega(n^\alpha)$ steps, we can build $E_{\alpha+1}$, an example in $\Omega(n^{\alpha+1})$ steps, based on the same principle as in Subsection 5.2, by adding a layer. If E_α has $L\beta$ processes $p_{(i,j)}$ ($1 \leq i \leq L, 1 \leq j \leq \beta$), then $E_{\alpha+1}$ has $L' = 2L$ lines of β processes $q_{(i',j')}$ ($1 \leq i' \leq L', 1 \leq j' \leq \beta$). The construction principle is as follows:

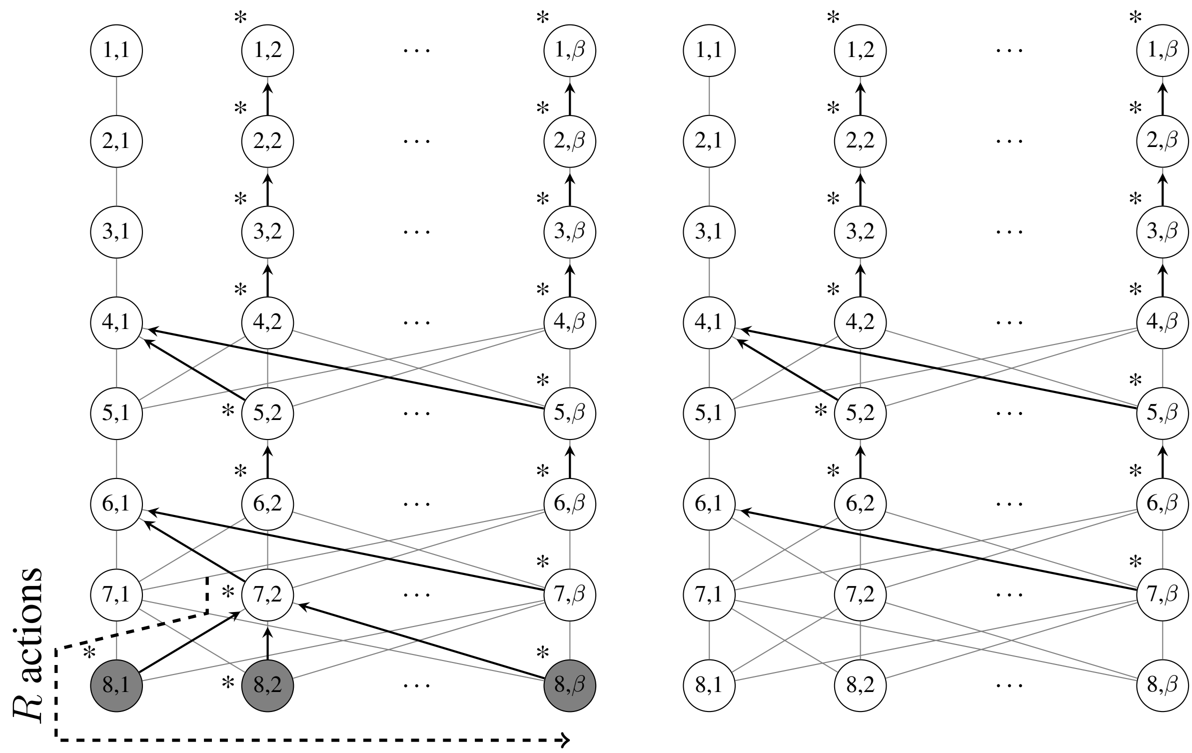
Algorithm 3 Algorithm of the daemon.

```
function DAEMON
    for i = 1 ... β, (i++) do
        RESET(3,i);
        if i < β then
            BUILD(3,i+1);
        end if
    end for
end function

function RESET(layer, column)
    for i = top(layer + 1) ... bottom(layer + 1), (i++) do
        p_(i,column) executes R;
    end for
    if layer = 1 then
        for j = 1 ... β, (j++) do
            p_(L,j) executes R;                    ▷ Reset of layer 1, L = top(1) = 8
        end for
    else
        for j = 1 ... β, (j++) do
            RESET(layer − 1, j);
            if j < β then
                BUILD(layer − 1, j + 1);
            end if
        end for
    end if
end function

function BUILD(layer, column)
    for i = top(layer) ... bottom(layer), (i++) do
        for j = 1 ... β, (j++) do
            p_(i,j) executes J;
        end for
        for k = i − 1 ... 2(i − L/2), (k−−) do
            if k ≥ top(layer) then
                for j = 1 ... β, (j++) do
                    p_(k,j) executes C1;
                end for
            else
                p_(k,column) executes C1;
            end if
        end for
        for k = i ... 2(i − L/2) + 1, (k−−) do
            if k ≥ top(layer) then
                for j = 1 ... β, (j++) do
                    p_(k,j) executes C2;
                end for
            else
                p_(k,column) executes C2;
            end if
        end for
    end for
    if layer > 1 then
        BUILD(layer − 1, 1);
    end if
end function
```



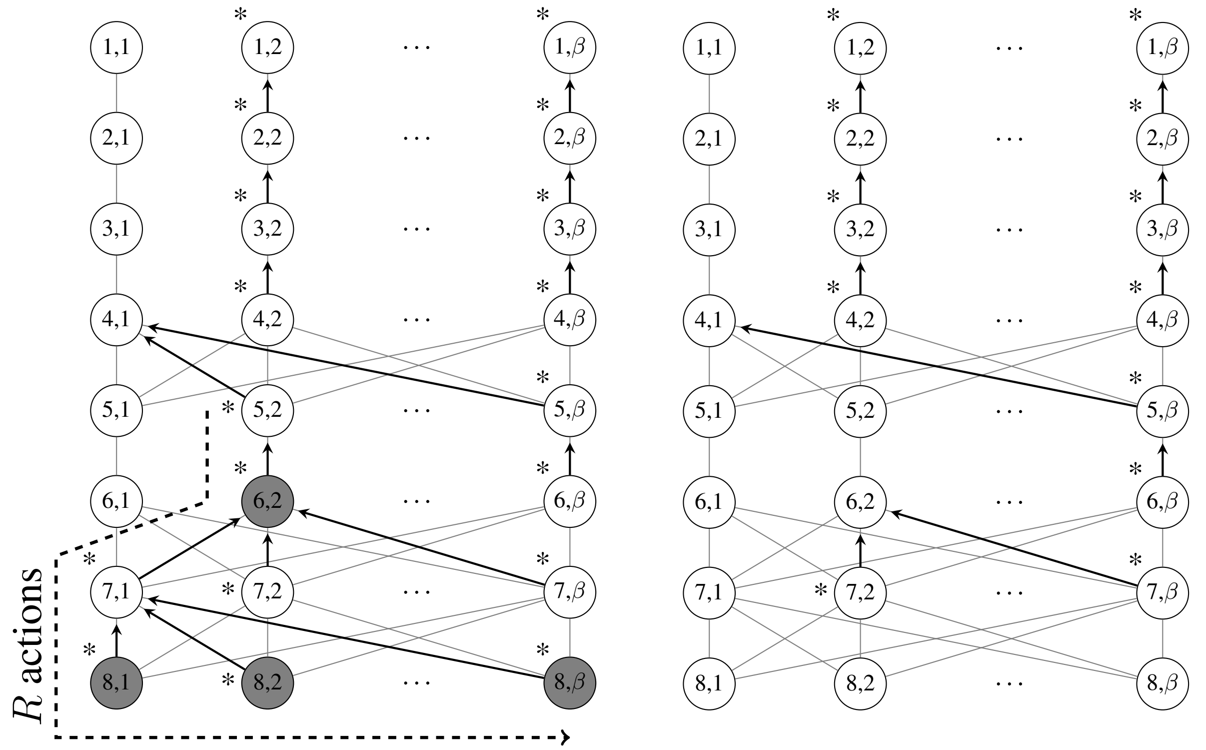
(a) Initial configuration. (b) Reset of the left column and the layer 1.

Figure 12: First sequence of actions R.



(a) BUILD(1,2) (b) Reset.

Figure 13: Reconstruction of the layer 1 on the second column and reset.



(a) BUILD(2,2) (b) Reset.

Figure 14: Reconstruction of the layer 2 on the second column and reset.

1. We increase the $level$ and the ID of the $L\beta$ processes of E_α as follows: $\forall i \in [1 \ldots L]$, $\forall j \in [1 \ldots \beta]$, $q_{(i+L,j)} = p_{(i,j)}$. The ID of $q_{(i+L,j)}$ becomes $(i + L - 1)\beta + j$ and $q_{(i+L,j)}.level = i + L$. The value of variables $color$ and $done$ do not change. If $i \neq 1$, the $parent$ remains the same. Otherwise, see step 3.
2. At the top of E_α, we add L lines of β processes. These new processes satisfy:
   - $\forall i \in [1 \ldots L], \forall j \in [1 \ldots \beta]$, $q_{(i,j)}.id = (i-1)\beta + j$, $q_{(i,j)}.leader = 0$, $q_{(i,j)}.level = i$ and $q_{(i,j)}.done = \texttt{false}$.
   - $\forall i \in [2 \ldots L], \forall j \in [1 \ldots \beta]$, $\{q_{(i-1,j)}, q_{(i,j)}\} \in V$ and $q_{(i,j)}.parent = q_{(i-1,j)}$.
   - $\forall j \in [1 \ldots \beta], q_{(1,j)}.parent = q_{(1,j)}$.
   - $\forall j \in [2 \ldots \beta], \forall i \in [1 \ldots L]$, $q_{(i,j)}.color = 2$.
   - $\forall i \in [1 \ldots L]$, $q_{(i,1)}.color = (i \mod 2) + 1$.
3. The former first line of E_α becomes a new bipartite complete subgraph with the last added line:
   - $\forall j \in [1 \ldots \beta], \forall j' \in [1 \ldots \beta]$, $\{q_{(L,j)}, q_{(L+1,j')}\} \in V$.
   - $\forall j \in [1 \ldots \beta]$, $q_{(L+1,j)}.parent = q_{(L,1)}$.

Figure 15 shows the structure of the network for E_5 and its initial configuration.

Then, the daemon selects processes according to function DAEMON(α+1) (see Algorithm 4) which is the generalization of the algorithm presented in section 5.2. In E_α, processes $p_{(L,.)}$ execute β^α times action R. Now, we added a new level of recursion. Then, processes $q_{(L',.)}$ execute $\beta^{\alpha+1}$ times action R. $\beta = \frac{n}{L'}$ hence the execution lasts $\Omega(n^{\alpha+1})$ steps. Hence, we obtain:

Theorem 13. For every $\alpha \geq 3$, for every $\beta \geq 2$, there exists a network $E_{\alpha+1}$ of $n = 2^{\alpha-3} \times 8 \times \beta$ processes in which there exists a possible execution that stabilizes in $\Omega(n^{\alpha+1})$ steps.

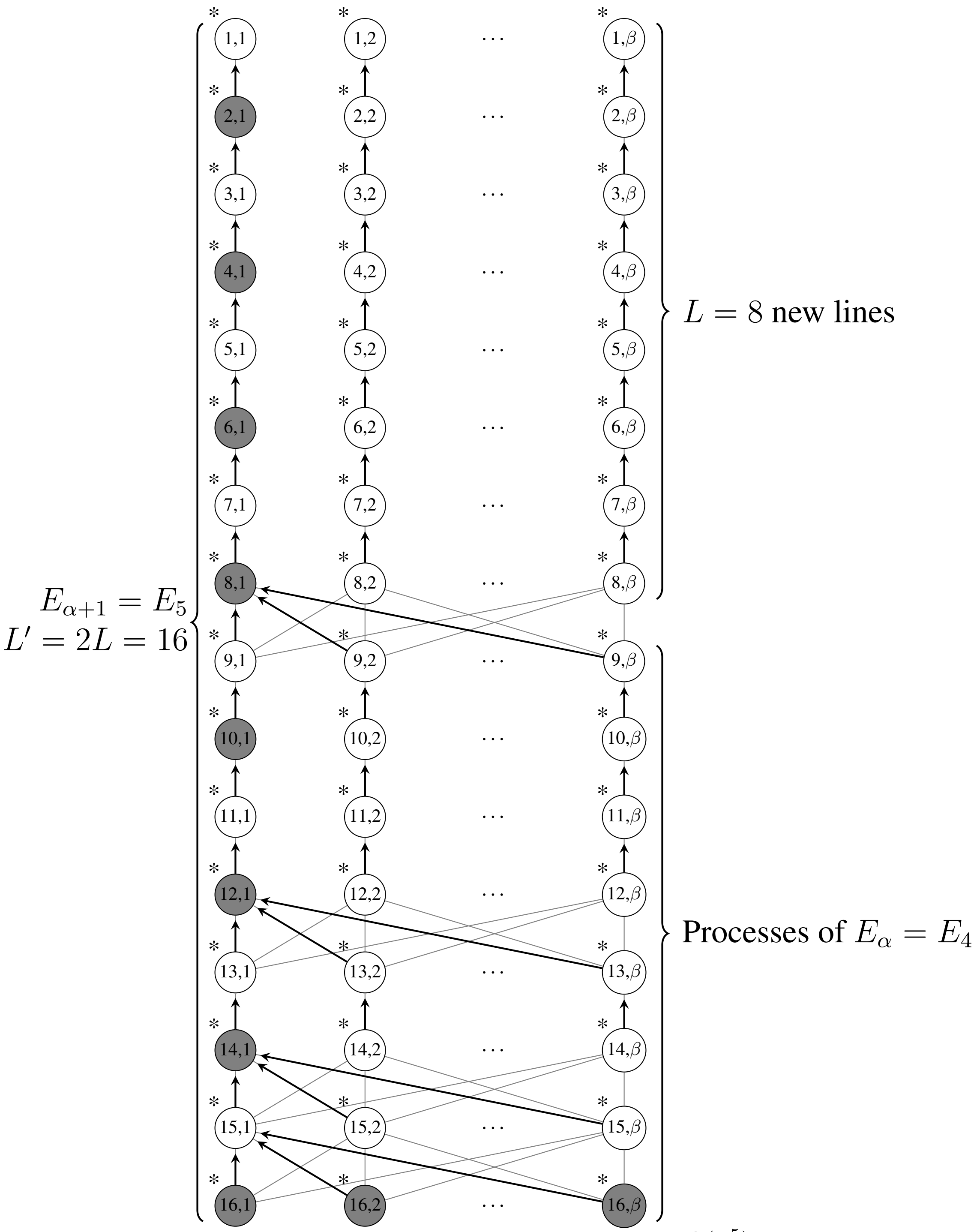


Figure 15: Initial configuration of the example in $O(n^5)$ steps.

Algorithm 4 Generalization of the algorithm of the daemon for $E_{\alpha+1}$.

```
function DAEMON(α + 1)
    for i = 1 . . . β, (i + +) do
        RESET(α,i);                                        ▷ See Algorithm 3
        if i < β then
            BUILD(α,i+1);                                  ▷ See Algorithm 3
        end if
    end for
end function
```

— 6 —
Conclusion

We proposed a silent self-stabilizing leader election algorithm, called $\mathcal{LE}$, for bidirectional connected identified networks of arbitrary topology. Starting from any arbitrary configuration, $\mathcal{LE}$ converges to a terminal configuration, where all processes know the ID of the leader, this latter being the process of minimum ID. Moreover, as in most of the solutions from the literature, a distributed spanning tree rooted at the leader is defined in the terminal configuration.

$\mathcal{LE}$ is written in the locally shared memory model. It assumes the distributed unfair daemon, the most general scheduling hypothesis of the model. Moreover, it requires no global knowledge on the network (such as an upper bound on the diameter or the number of processes, for example). $\mathcal{LE}$ is asymptotically optimal in space, as it requires $\Theta(\log n)$ bits per process, where n is the size of the network. We analyzed its stabilization time both in rounds and steps. We showed that $\mathcal{LE}$ stabilizes in at most $3n + \mathcal{D}$ rounds, where $\mathcal{D}$ is the diameter of the network. We also proved that for every $n \geq 4$, for every $\mathcal{D}, 2 \leq \mathcal{D} \leq n-2$, there is a network of n processes, in which a possible execution exactly lasts this complexity.

Finally, we proved that $\mathcal{LE}$ achieves a stabilization time polynomial in steps. More precisely, its stabilization time is at most $\frac{n^3}{2} + 2n^2 + \frac{n}{2} + 1$ steps. Then, we showed for every $n \geq 4$, that there exists a network of n processes, in which a possible execution exactly lasts $\frac{n^3}{6} + \frac{5}{2}n^2 - \frac{11}{3}n + 2$ steps, establishing then that the worst case is in $\Theta(n^3)$.

For fair comparison, we studied the step complexity of the previous best algorithm with similar settings (*i.e.*, it does not use any global knowledge and is proven assuming an unfair daemon) given in [9] and called here $\mathcal{DLV}$. We showed that for a given $\alpha \geq 3$, for every $\beta \geq 2$, there exists a network of $n = 2^\alpha \times \beta$ processes in which there is an execution that stabilizes in $\Omega(n^{\alpha+1})$. In other words, the stabilization time of $\mathcal{DLV}$ in steps is not polynomial.

We have also implemented $\mathcal{LE}$ in a high-level simulator to empirically evaluate its average performances. Experimental results tend to show that its worst case in terms of rounds ($\Theta(3n + \mathcal{D})$ rounds) is rare. Nevertheless, this work is still in progress. The experimentation protocol and some results are given in Appendix B.

Perspectives of this work deal with complexity issues. In [9], Datta *et al* showed that it is easy to implement a silent self-stabilizing leader election which works assuming an unfair daemon, uses $\Theta(\log n)$ bits per process, and stabilizes in $O(D)$ rounds (where D is an upper bound on $\mathcal{D}$), yet *if processes have knowledge of* D. Now, it is worth investigating if it is possible to design an algorithm which works assuming an unfair daemon, uses $\Theta(\log n)$ bits per process, and stabilizes in $O(\mathcal{D})$ rounds without using any global knowledge. We believe this problem remains difficult, even adding some fairness assumption.

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— A —

E_4 step by step

In this chapter , we detail an execution of $\mathcal{DLV}$ in E_4 following DAEMON (see Algo. 3) for any β (Listing A.1) and for $\beta = 3$ (Listing A.2). With this latter, we provide an "empty" representation of the network that can be used by the reader (Fig. 16).

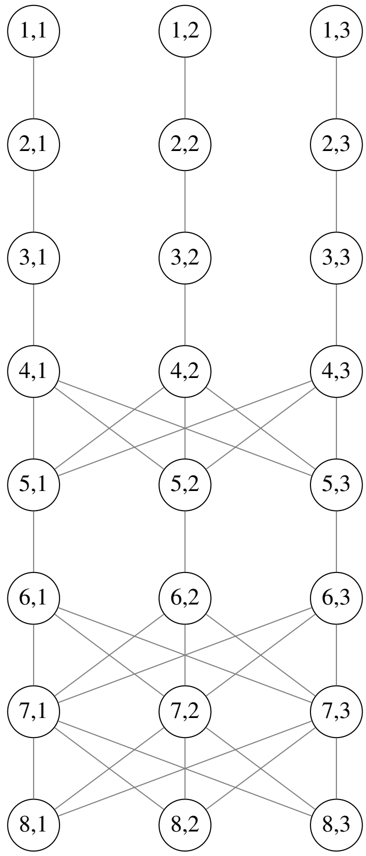

Figure 16: Empty representation of the network for E_4 with $\beta = 3$. The reader can use it to follow the step by step execution.

Listing A.1: Step by step execution of $\mathcal{DLV}$ in E_4 following DAEMON

```
//Reset(3,.) is called β times
Reset(3,1){
  //Reset(3,1) calls β times Reset(2,.)
  (1,1),(2,1),(3,1),(4,1) execute R
  Reset(2,1){
    //Reset(2,1) calls β times Reset(1,.)
    (5,1),(6,1) execute R
    Reset(1,1){
      //(8,.) executes β times R in Reset(1,1)
      (7,1) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    Build(1,2){
      (8,1),(8,2),...,(8,β) execute J
    }
    Reset(1,2){
      (7,2) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    (...)
    Build(1,β)
    Reset(1,β)
  }
  Build(2,2){
    (7,1),(7,2),...,(7,β) execute J
    (6,2) executes C1
    (7,1),(7,2),...,(7,β) execute C2
    Build(1,1){
      (8,1),(8,2),...,(8,β) execute J
    }
  }
  Reset(2,2){
    (5,2),(6,2) execute R
    Reset(1,1){
      (7,1) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    Build(1,2){
      (8,1),(8,2),...,(8,β) execute J
    }
    Reset(1,2){
      (7,2) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    (...)
    Build(1,β)
    Reset(1,β)
  }
  (...)
  Build(2,β)
  Reset(2,β)
}
Build(3,2){
  (5,1),(5,2),...,(5,β) execute J
  (4,2),(3,2),(2,2) execute C1
  (5,1),(5,2),...,(5,β) execute C2
  (4,2),(3,2) execute C2
  (6,1),(6,2),...,(6,β) execute J
  (5,1),(5,2),...,(5,β) execute C1
  (4,2) executes C1
  (6,1),(6,2),...,(6,β) execute C2
  (5,1),(5,2),...,(5,β) execute C2
  Build(2,1){
    (7,1),(7,2),...,(7,β) execute J
    (6,1) executes C1
    (7,1),(7,2),...,(7,β) execute C2
    Build(1,1){
      (8,1),(8,2),...,(8,β) execute J
    }
  }
}
Reset(3,2){
  (1,2),(2,2),(3,2),(4,2) execute R
  Reset(2,1){
    (5,1),(6,1) execute R
    Reset(1,1){
      (7,1) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    Build(1,2){
      (8,1),(8,2),...,(8,β) execute J
    }
    Reset(1,2){
      (7,2) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    (...)
    Build(1,β)
    Reset(1,β)
  }
  Build(2,2){
    (7,1),(7,2),...,(7,β) execute J
    (6,2) executes C1
    (7,1),(7,2),...,(7,β) execute C2
    Build(1,1){
      (8,1),(8,2),...,(8,β) execute J
    }
  }
  Reset(2,2){
    (5,2),(6,2) execute R
    Reset(1,1){
      (7,1) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    Build(1,2){
      (8,1),(8,2),...,(8,β) execute J
    }
    Reset(1,2){
      (7,2) executes R
      (8,1),(8,2),...,(8,β) execute R
    }
    (...)
    Build(1,β)
    Reset(1,β)
  }
  (...)
  Build(2,β)
  Reset(2,β)
}
(...)
Build(3,β)
Reset(3,β)
```

Listing A.2: Step by step execution of $\mathcal{DLV}$ in E_4 with $\beta = 3$ following DAEMON

(1,1) executes R
(2,1) executes R
(3,1) executes R
(4,1) executes R
(5,1) executes R
(6,1) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(7,1),(7,2),(7,3) execute J
(6,2) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(5,2) executes R
(6,2) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(7,1),(7,2),(7,3) execute J
(6,3) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(5,3) executes R
(6,3) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(5,1),(5,2),(5,3) execute J
(4,2) executes C1
(3,2) executes C1
(2,2) executes C1
(5,1),(5,2),(5,3) execute C2
(4,2) executes C2
(3,2) executes C2
(6,1),(6,2),(6,3) execute J
(5,1),(5,2),(5,3) execute C1
(4,2) executes C1
(6,1),(6,2),(6,3) execute C2
(5,1),(5,2),(5,3) execute C2
(7,1),(7,2),(7,3) execute J
(6,1) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(1,2) executes R
(2,2) executes R
(3,2) executes R
(4,2) executes R
(5,1) executes R
(6,1) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(7,1),(7,2),(7,3) execute J
(6,2) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(5,2) executes R
(6,2) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(7,1),(7,2),(7,3) execute J
(6,3) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(5,3) executes R
(6,3) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(5,1),(5,2),(5,3) execute J
(4,3) executes C1
(3,3) executes C1
(2,3) executes C1
(5,1),(5,2),(5,3) execute C2
(4,3) executes C2
(3,3) executes C2
(6,1),(6,2),(6,3) execute J
(5,1),(5,2),(5,3) execute C1
(4,3) executes C1
(6,1),(6,2),(6,3) execute C2
(5,1),(5,2),(5,3) execute C2
(7,1),(7,2),(7,3) execute J
(6,1) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(1,3) executes R
(2,3) executes R
(3,3) executes R
(4,3) executes R
(5,1) executes R
(6,1) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(7,1),(7,2),(7,3) execute J
(6,2) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(5,2) executes R
(6,2) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R
(7,1),(7,2),(7,3) execute J
(6,3) executes C1
(7,1),(7,2),(7,3) execute C2
(8,1),(8,2),(8,3) execute J
(5,3) executes R
(6,3) executes R
(7,1) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,2) executes R
(8,1),(8,2),(8,3) execute R
(8,1),(8,2),(8,3) execute J
(7,3) executes R
(8,1),(8,2),(8,3) execute R

— B —

Experimentation

In this appendix, we evaluate the average performances of algorithm $\mathcal{LE}$. We ran simulations on two different kinds of random graphs: Barabási-Albert graphs and Unit Disk Graphs. This work is still in progress.

B.1 Graph models

The *Barabási-Albert model* [2] generates random *scale-free* networks (*i.e.*, networks with a power-law degree distributions) similar to a lot of actual systems, the Internet for example. It models *preferential attachment*: a node with high degree receives new links with a bigger probability than a node with smaller degree.

In a *Unit Disk Graph (UDG)* [14], a node is connected with all the other nodes in a disk around it. In other words, two nodes are connected if and only if the Euclidean distance between them is smaller than some radius. Wireless sensor networks can be roughly modeled using UDG where the radius of the disk is the transmission range of the sensor emitter.

B.2 Experimentation Protocol

For each kind of graph, we generated a pool of ten random graphs of $n = 1000$ nodes for each value of the diameter (between 2 and 14 for Barabási-Albert graphs, between 4 and 27 for UDGs).

We use a simulator dedicated to locally shared memory model. We execute $\mathcal{LE}$ five times on each graph of the pool, until the confidence interval is smaller than 2% of the average stabilization time in round. The daemon used (for the moment) is randomly uniform: an enabled node is chosen with probability $\frac{1}{2}$.

The initialization of the processes is also randomized:

- The number of fake ids smaller than ℓ, denoted n_f is uniformly chosen between 0 and 10% of n.
- Each process has a unique random id between n_f and $n + n_f - 1$.
- n_f processes (uniformly chosen) have a random idR between 0 and $n_f - 1$. The other processes have a random idR between n_f and $n + n_f$.
- The par pointer is uniformly chosen among the neighbors of the node and itself.
- The $level$ is uniformly chosen between 0 and an arbitrary value.
- The $status$ is uniformly chosen.

B.3 Results

An experimental analysis was realized to evaluate the average performances of $\mathcal{LE}$ in terms of rounds. For example, Figure 17 shows experimental results for Barabási-Albert graphs. The average stabilization time in rounds is drastically smaller (the order of magnitude is the diameter) than the analytical bound in the worst case of $\Theta(3n + \mathcal{D})$ rounds. So the worst case seems to be rare in this class of graph. We speculate that the different behavior observed for small diameters (between 2 and 7) is a consequence of the high density of those graphs.

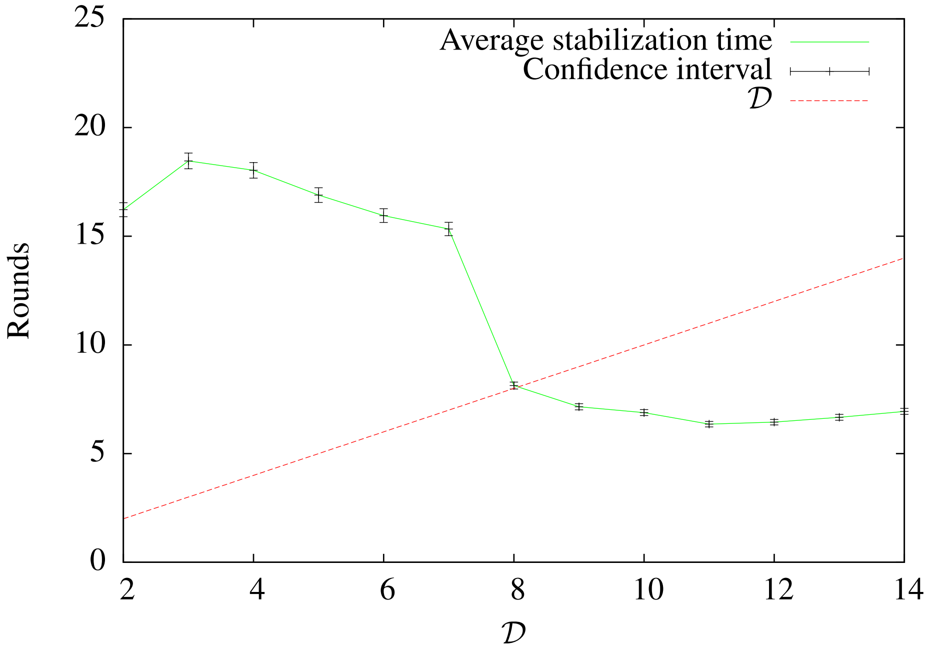

Figure 17: Average stabilization time in rounds for Barabási-Albert graphs ($n = 1000$).

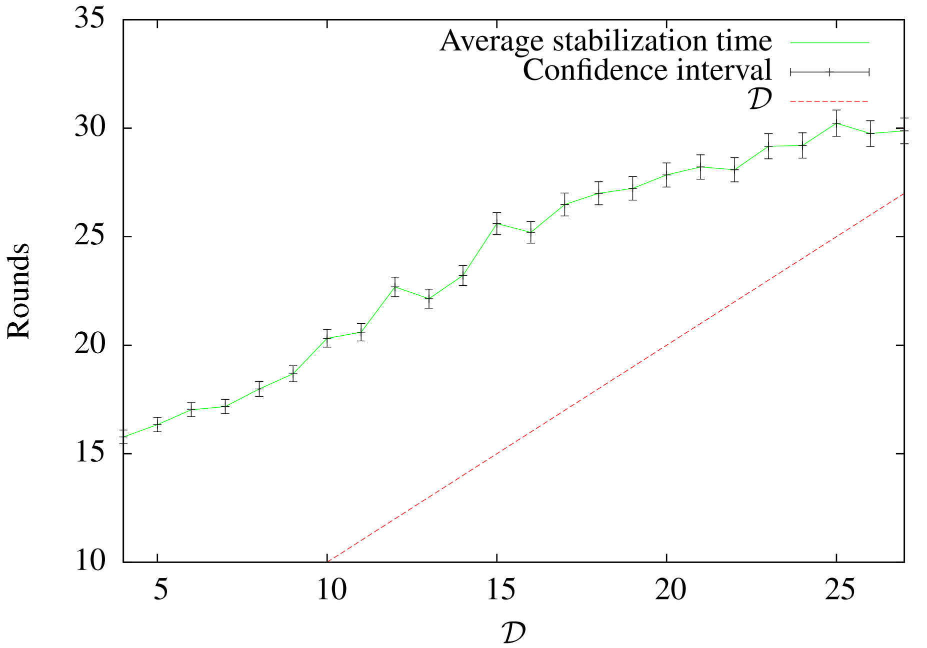

Figure 18: Average stabilization time in rounds for UDGs ($n = 1000$).

Nevertheless this conjecture requires much more investigation. The same conclusions ensue from the experimental results on UDGs (see Figure 18).

Other experiments have been done by inserting faults in a terminal configuration in order the impact of the number of faults on the stabilization time.

Again, this work needs further investigation, in particular using more realistic daemons.