Constante de Seshadri et Applications à l'approximation diophantienne (d'après D. Mckinnon et M. Roth)

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Abstract. — In this paper we study Diophantine approximation of rational points in projective varieties. In particular we see that the Seshadri constant, which is defined geometrically, has many Diophantine properties. We explain the proof of Roth type theorems for projective varieties based on the Faltings-Wüstholz theorem. As applications we state some simultaneous approximation theorems and deduce a special version of Vojta type inequality.

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CONTENTS

CHAPTER 1

PRELIMINARIES ON ALGEBRAIC GEOMETRY

In this chapter, X denotes an regular projective scheme of finite type over a field k of characteristic 0. In particular the Picard group $\mathrm{Pic}(X)$, the Weil divisor class group $A_{\dim X-1}(X)$, the Cartier divisor class group $\mathrm{CaCl}(X)$ are isomorphic ([**7**], II.6). We frequently use this identification without specifying it.

1.1. Blow-up

Let $\mathcal{I}$ be a coherent sheaf of ideals. With convention $\mathcal{I}^0 = \mathcal{O}_X$, The sheaf $\mathcal{F} = \bigoplus_{d=0}^{\infty} \mathcal{I}^d$ has a natural structure as a graded $\mathcal{O}_X$-algebra. On each affine open subset U of X, we associate the projective scheme structure $\mathbf{Proj}\,\mathcal{F}(U)$, and its natural morphism structural morphism $\pi_U : \mathbf{Proj}\,\mathcal{F}(U) \to U$. Gluing them together, we obtain a scheme $\widetilde{X}$ and a natural morphism $\pi : \widetilde{X} \to X$ such that for any affine subset U of X, $\pi^{-1}(U) \cong \mathbf{Proj}\,\mathcal{F}(U)$. Let Y be the closed subscheme of X corresponding to $\mathcal{I}$, $\widetilde{X}$ is called the *blow-up of X along Y*. The *exceptional divisor* E of $\widetilde{X}$ is defined to be the closed subscheme $\pi^{-1}(Y)$.

Let $f : Z \to X$ be morphism of schemes. We denote the image of the natural morphism $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Z$ by $f^{-1}(\mathcal{I})$, which is a coherent sheaf of ideals on Z. We have a morphism of schemes $\tilde{f} : \widetilde{Z} \to \widetilde{X}$ by the universal property of blow up ([**7**], II 7.14) where $\pi' : \widetilde{Z} \to Z$ is the blow up of Z with respect to the graded $\mathcal{O}_Z$-algebra generated by $f^{-1}(\mathcal{I})$. That is, we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\tilde{f}} & \widetilde{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{f} & X \end{array}$$

When Z is a closed subscheme of X which is not contained in the subscheme W of X defined by $\mathcal{I}$, then the scheme $\widetilde{Z}$ so obtained is a closed subscheme of $\widetilde{X}$. It is called the *strict transform* of Z. The exceptional divisor of $\widetilde{X}$ restricts to the exceptional divisor of $\widetilde{Z}$. Set-theoretically, the strict transform of Z is the Zariski closure of

$\pi^{-1}(Z - W)$ in $\widetilde{X}$. When $Z \cap W = \varnothing$, the strict transform of Z is isomorphic to Z. When W is a subscheme of Z, the strict transform of Z is the blow up along W (considered as a subscheme of Z defined by a coherent sheaf of ideals of $\mathcal{O}_Z$), which is also isomorphic to the fiber product $Z \times_X \widetilde{X}$.

Example 1.1.1 **(blow-up of a point)**. — We fix an embedding $X \hookrightarrow \mathbf{P}_k^n$. If we blow up $\mathbf{P}_k^n$ along a smooth k-point x of X, then the strict transform of X in $\widetilde{\mathbf{P}_k^n}$ is isomorphic to $\widetilde{X}$, which is the blow up of X along x, hence a closed subscheme of $\mathbf{P}_k^n \times_k \mathbf{P}_k^{n-1}$. If we denote the exceptional divisor of $\widetilde{\mathbf{P}_k^n}$ by D, then D is isomorphic to the projective space $\mathbf{P}_k^{n-1}$ and the exceptional divisor E of $\widetilde{X}$ is isomorphic to $\widetilde{X} \cap D$ (scheme theoretic intersection). Let $p : \mathbf{P}_k^n \times_k \mathbf{P}_k^{n-1} \to \mathbf{P}_k^n$ and $q : \mathbf{P}_k^n \times_k \mathbf{P}_k^{n-1} \to \mathbf{P}_k^{n-1}$ be projection maps.

Proposition 1.1.2. — The line bundle $\mathcal{O}_{\widetilde{X}}(E)$ is the restriction to X of the line bundle $\mathcal{O}_{\mathbf{P}_k^n \times_k \mathbf{P}_k^{n-1}}(1, -1) = p^*\mathcal{O}_{\mathbf{P}_k^n}(1) \otimes q^*\mathcal{O}_{\mathbf{P}_k^{n-1}}(-1)$.

Proof. — We need only to show this for $X = \mathbf{P}_k^n$ because $\mathcal{O}_{\widetilde{X}}(E) = \mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(D) \mid_{\widetilde{X}}$. Without loss of generality we assume $x = [1 : 0 : \cdots : 0]$. Note that in this case π is just the first projection p. x is contained the open set $U_{x_0} = \{x_0 \neq 0\} \cong \mathbf{A}_k^n$. The blow-up $\pi_{U_{x_0}} : \mathbf{Proj}\,\mathcal{F}_x(U_{x_0}) \to U_{x_0}$ is the closed subscheme of $\mathbf{A}_k^n \times \mathbf{P}_k^{n-1}$ defined by the homogeneous ideal $\{X_iY_j - X_jY_i\}_{1 \leqslant i,j \leqslant n}$ where X_i (resp. Y_i) stand for the coordinates of $\mathbf{A}_k^n$ (resp. $\mathbf{P}_k^{n-1}$). We can write down the blow-up equations on this open set:

$$\widetilde{U_{x_0}} = \{(x_1, \cdots, x_n) \times [y_1, \cdots, y_n] \in \mathbf{A}_k^n \times \mathbf{P}_k^n : x_iy_j = x_jy_i,\ 1 \leqslant i, j \leqslant n\}$$

On the open set $V_{y_n} = \{y_n \neq 0\}$, we have local coordinates $x_n, y_1, \cdots, y_n$ and V_{y_n} is defined by equations $x_i = x_ny_i$ $(1 \leqslant i \leqslant n)$. Choose the hyperplane $H = \{x_1 = 0\}$ containing x. Identify it with a section $s \in H^0(\mathbf{P}_k^n, \mathcal{O}_{\mathbf{P}_k^n}(1))$, $\pi^*s = x_ny_1 \in H^0(\widetilde{\mathbf{P}_k^n}, \mathcal{O}_{\mathbf{P}_k^n \times_k \mathbf{P}_k^{n-1}}(1, 0))$, this section defines two hyperplanes in $\widetilde{U_{x_0}}$. One is $\{x_n = 0\} = E$ and the other is $\{y_1 = 0\} = G$ which is the strict transform of H. And we see that G is the zero locus of a section of q^*t where $t \in H^0(\mathbf{P}^{n-1}, \mathcal{O}_{\mathbf{P}^{n-1}}(1))$. Similarly one verifies this relation on other principal open subsets. So we get that $\pi^*H = G + E$ as divisors. Hence

$$\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(E) \cong p^*\mathcal{O}_{\mathbf{P}_k^n}(1) \otimes q^*\mathcal{O}_{\mathbf{P}_k^{n-1}}(-1) = \mathcal{O}_{\mathbf{P}_k^n \times_k \mathbf{P}_k^{n-1}}(1, -1)|_{\widetilde{\mathbf{P}_k^n}}.$$

□

Corollary 1.1.3. — $\mathcal{O}_E(E) = \mathcal{O}_E(-1)$ and $(E^{\dim X}) = (-1)^{\dim X - 1} \deg E$ where $\deg E$ is the degree of E in $\mathbf{P}_k^{n-1}$.

Proof. — The first formula is clear from the proposition. Indeed, denote by $\mathfrak{I}(E)$ the coherent sheaf of ideals defining the reduced subscheme E. Then we have

$$\begin{aligned}\mathcal{O}_E(E) &= \mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(E)\mid_E\\ &= \mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(E)\otimes \mathcal{O}_{\widetilde{\mathbf{P}_k^n}}/\mathfrak{I}(E)\\ &= (p^*\mathcal{O}_{\mathbf{P}_k^n}(1)\otimes \mathcal{O}_{\widetilde{\mathbf{P}_k^n}}/\mathfrak{I}(E))\otimes (q^*\mathcal{O}_{\mathbf{P}_k^{n-1}}(-1)\otimes \mathcal{O}_{\widetilde{\mathbf{P}_k^n}}/\mathfrak{I}(E))\\ &= q^*\mathcal{O}_{\mathbf{P}_k^{n-1}}(-1)\otimes \mathcal{O}_{\widetilde{\mathbf{P}_k^n}}/\mathfrak{I}(E)\\ &= q^*\mathcal{O}_{\mathbf{P}_k^{n-1}}(-1)\mid_E\\ &= \mathcal{O}_E(-1).\end{aligned}$$

The second one follows from the fact that $((-E)^{\dim X}) = ((-E\mid_E)^{\dim X-1}) \cong ((\mathcal{O}_{\mathbf{P}_k^{n-1}}(1)\mid_E)^{\dim X-1})$ is the leading term of the Hilbert polynomial of E times $(\dim X-1)!$ which is equal to $\deg E$ (see [**3**] §1.2 for details). (Or in classical language, the degree of E is equal to the intersection number of $\dim E$ generic hyperplanes intersecting E.) □

For $m,\gamma\in\mathbf{N}$, let $H_m=\mathcal{O}_{\mathbf{P}_k^n}(m)$ and $L_{m,\gamma}=\pi^*H_m-\gamma E$.

Proposition 1.1.4. — The space of global sections of $L_{m,\gamma}$ can be identified with the space of hypersurfaces of degree m vanishing to order $\geqslant\gamma$ at point x.

Proof. — The line bundle $\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(-E)$ is equal to $\mathfrak{I}(E)$. And we have $\pi_*\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(-E)=\mathcal{I}_{\{x\}}\otimes\pi_*\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}$ where $\mathcal{I}_{\{x\}}$ is the coherent sheaf of ideals defining the reduced subscheme $\{x\}$ in $\mathbf{P}_k^n$. By Zariski's Main Theorem, $\pi_*\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}\cong\mathcal{O}_{\mathbf{P}_k^n}$. So $\pi_*\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(-E)=\mathcal{I}_{\{x\}}$. Using projection formula, we get

$$\pi_*\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(L_{m,\gamma})=\mathcal{O}_{\mathbf{P}_k^n}(m)\otimes\pi_*\mathcal{O}_{\widetilde{\mathbf{P}_k^n}}(-\gamma E)=\mathcal{O}_{\mathbf{P}_k^n}(m)\otimes\mathcal{I}_{\{x\}}^{\gamma}.$$

Hence $H^0(\widetilde{\mathbf{P}_k^n},L_{m,\gamma}) = H^0(\mathbf{P}_k^n,\mathcal{O}_{\mathbf{P}_k^n}(m)\otimes\mathcal{I}_{\{x\}}^{\gamma})$. Denote by x_γ the non-reduced subscheme $\{x\}$ with multiplicity γ. From the following exact sequence

$$0\longrightarrow\mathcal{O}_{\mathbf{P}_k^n}(m)\otimes\mathcal{I}_{\{x\}}^{\gamma}\xrightarrow{i}\mathcal{O}_{\mathbf{P}_k^n}(m)\xrightarrow{j}\mathcal{O}_{\mathbf{P}_k^n}(m)\mid_{x_\gamma}\longrightarrow 0$$

we get

$$0\longrightarrow H^0(\mathbf{P}_k^n,\mathcal{O}_{\mathbf{P}_k^n}(m)\otimes\mathcal{I}_{\{x\}}^{\gamma})\xrightarrow{i}H^0(\mathbf{P}_k^n,\mathcal{O}_{\mathbf{P}_k^n}(m))\xrightarrow{j}H^0(x_\gamma,\mathcal{O}_{\mathbf{P}_k^n}(m)\mid_{x_\gamma})\ .$$

So $H^0(\mathbf{P}_k^n,\mathcal{O}_{\mathbf{P}_k^n}(m)\otimes\mathcal{I}_{\{x\}}^{\gamma})=\operatorname{Ker} j$ is the set of sections in $H^0(\mathbf{P}_k^n,\mathcal{O}_{\mathbf{P}_k^n}(m))$ evaluated 0 to order $\geqslant\gamma$ at x. □

Remark 1.1.5. — Similarly this proposition holds for the blow up of $\mathbf{P}_k^n$ at finitely many closed points.

1.2. Nef/Big line bundles

Definition 1.2.1. — A line bundle L is said to be *numerically trivial* if for all irreducible curve $C \subset X$, we have

$$(L \cdot C) = \deg L|_C = 0.$$

The subgroup of $\mathrm{Pic}(X)$ generated by numerically trivial line bundles is denoted by $\mathrm{Pic}^0(X)$. The *Néron-Severi group* $N^1(X)$ is defined as $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. This is a finite generated abelian group. (To be more accurate and avoid confusion, the group defined here should be called "Numerical Equivalence Group" and is usually denoted by $\mathrm{Num}(X)$. It is a quotient group of the "true" Néron-Severi group, which is, by the traditional definition, the Picard group modulo algebraic equivalence. See [**7**] Chap. V. From now on we just adopt our definition above without specifying it.)

So $N^1(X)_{\mathbf{R}} = N^1(X) \otimes_{\mathbf{Z}} \mathbf{R}$ is a finite dimensional $\mathbf{R}$-vector space and we equipped it with the usual topology (fixing a base).

Definition 1.2.2. — A line bundle H on X is called *nef* if for any integral curve Y on X,

$$H \cdot Y = (H|_Y) = \deg(H|_Y) \geqslant 0.$$

***Theorem 1.2.3* (Kleiman).** — *Let H be a nef line bundle on X. The following statements are true:*

1. *For any integral subscheme Z of X,*
$$H^{\dim Z} \cdot Z = (H|_Z)^{\dim Z} \geqslant 0.$$
2. *H is ample if and only if*
$$(H^{\dim X}) > 0.$$

Definition 1.2.4. — Let L be a line bundle on X. The *volume* of L is defined as

$$\mathrm{Vol}(L) = \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^{\dim X}/(\dim X)!}.$$

The volume of a Cartier divisor is defined to be the volume of its corresponding line bundle.

L is called *big* if $\mathrm{Vol}(L) > 0$.

***Theorem 1.2.5* (Homotheticity of $\mathrm{Vol}(\cdot)$).** — *For any big line bundle L and $n \in \mathbf{N}_{>0}$, we have*

$$\mathrm{Vol}(nL) = n^{\dim X}\,\mathrm{Vol}(L).$$

From this theorem we can extend $\mathrm{Vol}(\cdot)$ to $\mathbf{Q}$-line bundles.

Theorem 1.2.6. — *If two Cartier divisors D and E are numerically equivalent (that is, they are in the same class of $N^1(X)$), then*

$$\mathrm{Vol}(D) = \mathrm{Vol}(E).$$

This makes $\mathrm{Vol}(\cdot)$ a well-defined function on $N^1(X)_{\mathbf{Q}}$.

Theorem 1.2.7. — *The function* Vol(·) *is continuous in the following sense. Fix a norm on* $N^1(X)_{\mathbf{R}}$ *inducing the usual euclidean topology. Then there exists a constant* $C > 0$ *such that for any* $D, E \in N^1(X)_{\mathbf{Q}}$*, we have*

$$|\operatorname{Vol}(D) - \operatorname{Vol}(E)| \leqslant C\left((\max \|D\|, \|E\|)\right)^{\dim X - 1} \|D - E\|.$$

So the function Vol(·) extends uniquely to a continuous function on $N^1(X)_{\mathbf{R}}$.

Definition 1.2.8. — The *nef cone* $\operatorname{Nef}(X)$(resp. *big cone* $\operatorname{Big}(X)$, *ample cone* $\operatorname{Amp}(X)$) of X is the set of all finite linear combinations of equivalent classes of nef (resp. big, ample) line bundles with positive coefficients. The *pseudoeffective cone* $\overline{\operatorname{Eff}}(X)$ is the closure of the convex hull spanned by equivalent classes of effective line bundles, all these cones are considered as subsets of $N^1(X)_{\mathbf{R}}$.

***Theorem 1.2.9* (Kleiman)**. — *The following statements are true:*

1. *The nef cone is the closure of ample cone, and the ample cone is the interior of the nef cone:*
$$\operatorname{Nef}(X) = \overline{\operatorname{Amp}(X)};\ \operatorname{Amp}(X) = \operatorname{int}(\operatorname{Nef}(X)).$$
2. *The pseudoeffective cone is the closure of big cone, and the big cone is the interior of the pseudoefficive cone:*
$$\overline{\operatorname{Eff}}(X) = \overline{\operatorname{Big}(X)};\ \operatorname{Big}(X) = \operatorname{int}(\overline{\operatorname{Eff}}(X)).$$

Since any ample line bundle is big (this is because some power of it is a very ample line bundle, a very ample line bundle is obtained by pulling back the universal line bundle on a projective space, hence clearly big), it follows from Kleiman's theorem that the nef cone is contained in the pseudoeffective cone.

The following theorem gives a characterization of big line bundles. They are in fact sums of ample and effective line bundles.

Theorem 1.2.10. — *A line bundle H on X is big if and only if for any ample line bundle L on X, there exists a integer $m > 0$ and an effective divisor E such that mH is linearly equivalent to $L + E$.*

The following theorem establishes a relationship between volume and self intersection number.

***Theorem 1.2.11* (Asymptotic Riemann-Roch)**. — *Let L be a nef line bundle on X. $n = \dim X$. Then*

$$h^0(X, L^{\otimes m}) = \frac{(L^n)}{n!} m^n + O(m^{n-1}).$$

In particular this theorem implies that if H is an ample line bundle, then

$$\operatorname{Vol}(H) = \lim_{m \to \infty} \frac{h^0(X, L^{\otimes m})}{m^{\dim X}/(\dim X)!} = (H^n).$$

For the proof of all these theorems, see ([**9**], Chap. 1,2).

1.3. Seshadri constant

We first quote a famous and classical criterion for ampleness, for a proof see ([**3**] §1.5).

***Theorem 1.3.1* (Ampleness criterion).** — *Let H be a line bundle on X. The following statements are equivalent:*

1. *H is ample;*
2. *(Nakai-Moishezon criterion) For any integral subscheme Y of X, we have*

$$(H^{\dim Y} \cdot Y) = ((H|_Y)^{\dim Y}) > 0;$$

3. *(Seshadri criterion) For any closed point $x \in X$, there exists a strictly positive constant $\varepsilon(x)$ (depending on x) such that for any integral 1-dimensional subscheme Z of X containing x, we have*

$$(H \cdot Z) \geqslant \varepsilon(x) \operatorname{mult}_x(Z).$$

This theorem says that ampleness can be verified through restriction to curves. (One should be aware that it is in general not sufficient to only verify ampleness through intersection number on all curves. It should be verified on all elements of the set $\overline{\mathrm{NE}}(\mathrm{X})$, the closure of "the cone of curves". See [**3**].)

Corollary 1.3.2. — Let L be an ample line bundle on X. Then for any $H \in \mathrm{Pic}^0(X)$, $L + H$ is ample.

Thus ampleness descends to the Néron-Severi group $N^1(X)$.

Inspired by the theorem, we define the following invariant to characterize the local positivity of a line bundle.

Definition 1.3.3. — Let H be a nef line bundle on X and $x \in X(\bar{k})$ a closed point. The number

$$\varepsilon(H, x) = \inf_{x \in C} \frac{(H \cdot C)}{\operatorname{mult}_x C},$$

where the infimum is taken through all integral curves passing through x, is called the *Seshadri constant* of H at x. Note that since H is nef, $\varepsilon(H, x)$ is always non-negative.

Clearly the computation of the Seshadri constant also descends to $N^1(X)$. We can also naturally extend this notion to any nef $\mathbf{Q}$-line bundles.

Proposition 1.3.4. — $\varepsilon(H, x)$ is largest real number γ such that the line bundle $H_\gamma = \pi^* H - \gamma E$ lies in the nef cone of $\widetilde{X}$ where $\pi : \widetilde{X} \to X_{\bar{k}}$ is the blow up of $X_{\bar{k}}$ along the point x.

Proof. — We choose an embedding of X into some projective space $\mathbf{P}^n_{\bar{k}}$. Then $\widetilde{X}$ is the strict transform of X in $\widetilde{\mathbf{P}^n_{\bar{k}}}$. Let $E' \cong \mathbf{P}^{n-1}_{\bar{k}}$ denote the exceptional divisor of $\widetilde{\mathbf{P}^n_{\bar{k}}}$ and let E denote the exceptional divisor of $\widetilde{X}$. Then E is the scheme theoretic intersection $\widetilde{X} \cap E'$. Suppose that $\gamma \in]0, \varepsilon(H,x)[$. For any integral curve $C \subset \widetilde{X}$, if $C \subset E$, then $H_\gamma \cdot C = \gamma \mathcal{O}_{E'}(1)|_C = \gamma \deg C$ (degree of C in E'). If $C \not\subset E$,

$$H_\gamma \cdot C = H \cdot \pi(C) - \gamma \operatorname{mult}_x \pi(C) \geqslant \begin{cases} (\varepsilon(H,x) - \gamma) \operatorname{mult}_x \pi(C), & x \in \pi(C); \\ H \cdot \pi(C), & x \notin \pi(C). \end{cases}$$

Since H is assumed to be nef, $H_\gamma \cdot C$ is always non-negative. Thus we have shown that H_γ is nef, hence $H_{\varepsilon(H,x)}$ is nef.
Conversely, for any $\eta > \varepsilon(H,x)$, from the definition of $\varepsilon(H,x)$ we can find an integral curve $D \subset X$ such that $\varepsilon(H,x) < H \cdot D < \eta \operatorname{mult}_x X$. Let $\widetilde{D}$ be the strict transform of D in $\widetilde{X}$. Then $H_\eta \cdot \widetilde{D} = H \cdot D - \eta \operatorname{mult}_x X < 0$ which implies that H_η is not nef. □

Corollary 1.3.5. — For any ample line bundle L and closed point $x \in X$, we have

$$\varepsilon(L,x) \leqslant \sqrt[n]{\frac{\operatorname{Vol}(L)}{\operatorname{mult}_x X}}$$

where $n = \dim X$.

Proof. — By taking some power of L, we may assume L is very ample. Then $(L^n) = \operatorname{Vol}(L)$. We still use notations from the proof above. Then we have

$$\deg(E) = \operatorname{mult}_x X;\ \pi^* L|_E \cong \mathcal{O}_E;\ \mathcal{O}_{\widetilde{X}}(E)|_E = \mathcal{O}_{\widetilde{X}}(E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(-1).$$

For any pair of integers i, j such that $i + j = n, j > 0$,

$$\begin{aligned} \pi^* L^i \cdot E^j &= (\pi^* L|_E)^i \cdot (\mathcal{O}_{\widetilde{X}}(E)|_E)^{j-1} \\ &= (\mathcal{O}_E)^i \cdot \mathcal{O}_E(-1)^{j-1} \\ &= \begin{cases} 0, & \text{if } i > 0, j > 0; \\ (-1)^{n-1} \deg E, & \text{if } i = 0, j = n \end{cases} \end{aligned}$$

using Proposition 1.1.2.For any $\gamma > \sqrt[n]{\frac{\operatorname{Vol}(L)}{\operatorname{mult}_x X}}$,

$$((\pi^* L - \gamma E)^n) = \sum_{i+j=n} (-\gamma)^j \pi^* L^i \cdot E^j = (L^n) - \gamma^n \operatorname{mult}_x X < (L^n) - (L^n) = 0.$$

So $\varepsilon(L,x) \leqslant \sqrt[n]{\frac{\operatorname{Vol}(L)}{\operatorname{mult}_x X}}$. □

We list more properties of Seshadri constant below.

Suppose X_1, X_2 are two schemes satisfying the assumption in the beginning of this chapter, and $x_i \in X_i(\bar{k})$. Let L_i be nef line bundles on X_i. We denote by $L_1 \boxtimes L_2$ the nef line bundle on $X_1 \times X_2$ which is equal to $p_1^* L_1 \otimes p_2^* L_2$ where $p_i : X_1 \times X_2 \to X_i$ is the i-th projection.

Proposition 1.3.6. — The following statements are true.

1. For any $x \in \mathbf{P}^n_k(\bar{k})$, $\varepsilon(\mathcal{O}(1), x) = 1$.
2. If H is ample, then $\varepsilon(H,x) > 0$.

3. $\varepsilon(\cdot, x)$ is a concave function on the nef cone. That is, for any nef **Q**-line bundles H_1, H_2 on X, and for any $a, b \in \mathbf{Q}_{\geqslant 0}$, we have
$$\varepsilon\left(aH_1 + bH_2), x\right) \geqslant a\varepsilon(H_1, x) + b\varepsilon(H_2, x).$$
4. $\varepsilon(L_1 \boxtimes L_2, x_1 \times x_2) = \min(\varepsilon(L_1, x_1), \varepsilon(L_2, x_2))$.

The proof of these statements is straightforward using the definition and the equivalent description. For details we refer to the original paper [**10**].

1.4. Heights

Definition 1.4.1. — Let $X \hookrightarrow \mathbf{P}^n_k$ be a projective scheme over a field k. A *height function* H is a map from $X(\bar{k})$ to $\mathbf{R}_{>0}$. Two height functions H, H' are said to be *equivalent* if there exists constants $C, c > 0$ such that
$$cH \leqslant H' \leqslant CH.$$
They are said to be *quasi-equivalent* if for any $\varepsilon > 0$, there exists two positive constants $c(\varepsilon)$ and $C(\varepsilon)$ such that
$$c(\varepsilon) \cdot H^{1-\varepsilon} \leqslant H' \leqslant C(\varepsilon) \cdot H^{1+\varepsilon}.$$

The use of height functions is to measure the complexity of rational points. For any projective space $\mathbf{P}^n_k$ over k, We define the (normalized) *absolute Weil height* to be the map $H_{\mathcal{O}(1)} : X(\bar{k}) \to \mathbf{R}$ as follows. For $x = [x_0 : \cdots : x_n] \in X(\bar{k})$, we choose a field K such that $\mathrm{Spec}(\bar{k}) \to X$ factorizes as $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(K) \to X$. We define
$$H_{\mathcal{O}(1)}([x_0 : \cdots : x_n]) = \prod_{v \in \mathcal{M}_K} \max_{i=0,\cdots,n} |x_i|_v.$$
Here we use the normalized absolute value:
$$|y|_v = |\operatorname{Norm}_{K_v/\mathbf{Q}_w}(y)|_w^{\frac{1}{[K:\mathbf{Q}]}}$$
where $w \in \mathcal{M}_{\mathbf{Q}}$ such that $v|w$. $H_{\mathcal{O}(1)}$ is a well-defined function on $\mathbf{P}^n_k(\bar{k})$. Indeed, we have the product formula:
$$\prod_{v \in \mathcal{M}_L} |y|_v = 1,\ \forall y \in L^*$$
for any number field L. Then if K' is also a field of definition for x, we have
$$\prod_{v \in \mathcal{M}_K} \max_{i=0,\cdots,n} |x_i|_v = \prod_{w \in \mathcal{M}'_K} \max_{i=0,\cdots,n} |x_i|_w.$$
Moreover, for any $\lambda \in \bar{k}^*$,
$$H_{\mathcal{O}(1)}([\lambda x_0, \cdots, \lambda x_n]) = H_{\mathcal{O}(1)}([x_0, \cdots, x_n]).$$

Theorem 1.4.2. — *There is a unique way to associate an equivalence class of height functions $H_L : X(k) \to \mathbf{R}$ to every element $[L]$ in* $\mathrm{Pic}(X)$*, such that the following properties hold:*

1. *For all* $[L]$ *and* $[L']$ *in* $\mathrm{Pic}(X)$, $H_{L\otimes L'} = H_L H_{L'}$ *as an equivalence class of height functions;*
2. *If we have an embedding* $\varphi : X \hookrightarrow \mathbf{P}^n_k$, *then for* $L = \varphi^*(\mathcal{O}_{\mathbf{P}^n_k}(1))$ *we have* $H_L(x) = H_{\mathcal{O}_{\mathbf{P}^n_k}(1)}(\varphi(x)) = H_{\mathcal{O}_{\mathbf{P}^n_k}(1)}([x_0 : \cdots : x_n]) = \prod_v \max_{i=0,\cdots,n} |x_i|_v$.

Furthermore, this definition extends multiplicatively to **Q***-line bundles.*

For a proof, see ([**12**] §2.8). In this paper, we only consider this kind of height functions.

Let L be a line bundle on X. We denote by $\varphi : X \to \operatorname{Spec} k$ the structural morphism.

Definition 1.4.3. — The *base locus* of L is the subscheme defined by the coherent sheaf of ideals $\mathfrak{b}$ which is determined by the image of the evaluation map

$$\varphi^*\varphi_* L \otimes_{\mathcal{O}_X} L^\vee \to \mathcal{O}_X$$

In other words, for any open affine subscheme $U = \operatorname{Spec} A$ of X such that L is trivialized on U, we can find some $s \in \Gamma(U, L)$ such that $L \cong s\mathcal{O}_X|_U$. For any $t \in \Gamma(X, L)$, it corresponds to some $f_t \in A$. Then on U the coherent sheaf of ideals $\mathfrak{b}$ is the ideal of A generated by $\{f_t, t \in \Gamma(X, L)\}$ and the base locus of L is the closed subscheme $\operatorname{Spec}(A/\mathfrak{b}(U))$.

Proposition 1.4.4. — If k is a number field (or more generally, any field satisfying product formula), then for any effective **Q**-line bundle L on X (which means that $\exists m > 0, L^{\otimes m}$ is effective), there exists a constant $C > 0$, such that $H_L(x) \geqslant C$ for all $x \in X \smallsetminus Ba(L)(\bar{k})$, where $Ba(L)$ denotes the asymptotic base locus of L.

Proof. — Since the height functions are multiplicative, we can assume L is effective. We denote the base locus of L by $Bs(L)$. By definition,

$$Ba(L) = \bigcap_{m>0} Bs(L^{\otimes m})_{\mathrm{red}}. \text{ (intersection as sets)}$$

$(M_n = \cap_{m=1}^n Bs(L^{\otimes m})_{\mathrm{red}})$ is a sequence of decreasing closed subsets of X. By noetherianity, $\exists m_0 > 0$ such that $\forall n \geqslant m_0$, $M_n = M_{m_0}$. Let $t = \prod_{m=1}^{m_0} m$. We claim that $L^{\otimes t}$ is base point free on $X \smallsetminus Ba(L)$. This gives a morphism

$$j : X \smallsetminus Ba(L) \hookrightarrow \mathbf{P}_k(\Gamma(X, L^{\otimes t})).$$

In fact, for any $m \leqslant m_0$, for $u \in \Gamma(X, L^{\otimes m})$, $u^{\otimes t/m} \in \Gamma(X, L^{\otimes t})$. Hence u vanishes on $Bs(L^{\otimes t})_{\mathrm{red}}$. So $Bs(L^{\otimes m})_{\mathrm{red}} \supset Bs(L^{\otimes t})_{\mathrm{red}}$. It follows that

$$Ba(L) = \cap_{m=1}^{m_0} Ba(L^{\otimes m})_{\mathrm{red}} \supset Bs(L^{\otimes t})_{\mathrm{red}}.$$

We choose a very ample line bundle H on X, which gives the embedding $i : X \hookrightarrow \mathbf{P}^r_k$, and we still denote its restriction to $X \smallsetminus Ba(L)$ by H. Then the line bundle $H \otimes L^{\otimes t}$ gives the following embedding

$$j \times i : \; X \smallsetminus Ba(L) \hookrightarrow \mathbf{P}_k(\Gamma(X, L^{\otimes t})) \times \mathbf{P}^r_k \hookrightarrow \mathbf{P}^s_k$$

where $s = l + r + lr,\ l = \dim(\Gamma(X, L^{\otimes t}))$. And we have $j^*\mathcal{O}(1) \cong H \otimes L^{\otimes t}$. For any $(x_i)_{i=0}^{l}, (y_j)_{j=0}^{r}$ not all zero, we suppose $x_0 \neq 0$. Then

$$H_{\mathcal{O}_{\mathbf{P}^s}(1)}(x_0y_0, \cdots, x_ly_r) \geqslant H_{\mathcal{O}_{\mathbf{P}^r}(1)}(x_0y_0, \cdots, x_0y_r) = H_{\mathcal{O}_{\mathbf{P}^r}(1)}(y_0, \cdots, y_r).$$

It follows that $H_{L^{\otimes t}} = H_{\mathcal{O}_{\mathbf{P}^s}(1)}/H_{\mathcal{O}_{\mathbf{P}^r}(1)} \geqslant 1$. So we get $H_L = (H_{L^{\otimes t}})^{1/t} \geqslant 1$. □

Corollary 1.4.5. — Moreover assume that L is ample, then there exists a constant $C > 0$, such that $H_L(x) \geqslant C$ for all $x \in X(\bar{k})$.

Proposition 1.4.6. — Let L_1, L_2 be line bundles on X such that L_1 is ample and L_2 is numerically trivial. Then $H_{L_1 \otimes L_2}$ and H_{L_1} are quasi-equivalent.

Proof. — Fix $\varepsilon > 0$. For any $n > \varepsilon^{-1}$, $L_2^{\otimes n}$ is numerically trivial (Corollary 1.3.2), so $L_1 \otimes L_2^{\otimes n}$ is ample. Therefore $\exists C(n) > 0$, such that $H_{L_1} H_{L_2}^n > C(n)$. So

$$H_{L_1 \otimes L_2} > C(n)^{\frac{1}{n}} H_{L_1}^{1-\frac{1}{n}} > C(n)^{\frac{1}{n}} H_{L_1}^{1-\varepsilon}$$

because H_{L_1} is positive since L_1 is ample. Similarly choosing $m < -\varepsilon^{-1}$, We get that $\exists c(m) > 0$ such that

$$H_{L_1 \otimes L_2} < c(m)^{\frac{1}{m}} H_{L_1}^{1-\frac{1}{m}} < c(m)^{\frac{1}{m}} H_{L_1}^{1+\varepsilon}.$$

□

CHAPTER 2

CLASSICAL APPROXIMATION THEOREMS OF ALGEBRAIC NUMBERS

References for this chapter are [**8**] and [**2**].

Throughout this chapter, whenever we mention phrases of the form "rational number p/q" we always mean that p, q are integers.

2.1. Irrationality measure

Definition 2.1.1. — Let z be a real number, a positive real number ν is called an *irrationality measure* for z if there exist a strictly positive constant c such that

$$\left|z - \frac{p}{q}\right| \geqslant \frac{c}{q^{\nu}}$$

for any rational number p/q with $q > 0$ distinct from z.

Roughly speaking, irrationality measure measures how well we can approximate a real algebraic number by rational numbers.

Definition 2.1.2. — We set the *approximation exponent* $\mu(z)$ of z to be the infimum of the set of all irrationality measures of z. Equivalently, $\mu(z)$ is the infimum of the set of all positive real numbers r such that the equation

$$\left|z - \frac{p}{q}\right| \leqslant \frac{1}{q^{r}}$$

has at most finitely many rational solutions p/q.

Remark 2.1.3. — With notation as in the definition above, if moreover there exists infinitely many rational numbers u/v with $v > 0$ such that

$$\left|z - \frac{u}{v}\right| < \frac{1}{v^{\nu}},$$

then any irrationality measure of z is not smaller than ν, i.e. $\mu(z) \geqslant \nu$.

2.2. Dirichlet's theorem

Using a simple "pigeon-hole" argument, Dirichlet proved the following theorem.

Theorem 2.2.1. — *The following statements are true:*

1. *Let ξ be an irrational number, then there exists infinitely many rational numbers p/q such that*
$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$
2. *Let a/b be a rational number, then for any rational number p/q with $q > 0$ distinct from a/b,*
$$\left|\frac{a}{b} - \frac{p}{q}\right| \geqslant \frac{1}{|b|q}.$$

Hence 1 is an irrationality measure for any rational number and any irrational measure of an irrational number is greater than 2. Furthermore it is easy to see that the approximation exponent of any rational number is exactly 1.

2.3. Liouville's theorem

Theorem 2.3.1. — *Let z be a real algebraic number of degree $d \geqslant 2$, then there exists a positive constant c such that for any rational number p/q,*
$$\left|z - \frac{p}{q}\right| \geqslant \frac{c}{q^d}.$$

This theorem gives way to construct uncountably many transcendent numbers. On the other hand it implies that for any real algebraic number of degree d, d is an irrationality measure, hence $\mu(z) \leqslant d$. This gives an upper bound for the approximation exponent.

Proof. — The proof is based on elementary arguments from calculus. First let $f(X) \in \mathbf{Z}[X]$ be the minimal polynomial of z. Then since $d \geqslant 2$, $f(p/q) \neq 0$ for any rational number p/q otherwise $f(X)$ would be reducible. So $|f(p/q)| \geqslant 1/q^d$. On the other hand, by mean value theorem, there exists a constant $c(z)$ such that $|f(p/q)| \leqslant c(z)|p/q - z|$. Then Liouville's theorem follows. □

2.4. Roth's theorem

Theorem 2.4.1. — *Let z be a real algebraic number of degree $d \geqslant 2$. Then for any $\varepsilon > 0$, there exists a strictly positive constant $c = c(\varepsilon)$ such that for any rational number p/q with $q > 0$,*
$$\left|z - \frac{p}{q}\right| > \frac{c(\varepsilon)}{q^{2+\varepsilon}}.$$

Combining Dirichlet's theorem, Roth's theorem implies the surprising fact that the approximation exponent of any irrational real algebraic number is exactly 2.

An outline of the proof of Roth's theorem Assume the contrary. Then as $\varepsilon \to 0$, we can always get sequence of rational numbers $\{p_i/q_i\}$ satisfying $|p_i/q_i - z| \leqslant 1/q_i^{2+\varepsilon}$. Roth constructed an auxiliary polynomial of multi variables (now called Roth's lemma) $f(X_1, \cdots, X_n)$ of degree $d_1, \cdots, d_n$ satisfying the following properties:

1. $f(p_1/q_1, \cdots, p_n/q_n)$ are either $\neq 0$ or vanish of small degree;
2. f vanishes at z at a very large degree.

Of course one still needs really hard extra work (e.g using a form of lemma due to Siegel to bound the coefficients of f). By taking certain derivatives of f (which does not affect much to the estimation), as in the proof of Liouville's theorem, we have the type of inequality: $|f(p_1/q_1, \cdots, p_n/q_n)| \geqslant 1/\prod q_i^{d_i}$. It means that f cannot be too small at those rational numbers. On the other hand, since these rational numbers are very close to z, f cannot be too big. This achieves a contradiction.

2.5. A simultaneous approximation version of Roth's theorem for $\mathbf{P}^1$

Theorem 2.5.1. — *Let K be a number field, $x \in \mathbf{P}^1_K(\bar{K})$, $S \subset \mathcal{M}(K)$ be a finite set of places of K. Assume that every place in S is extended to $\bar{K}$ in some way. Then for any $\varepsilon > 0$, there exist only finitely many solutions y to*

$$\prod_{v \in S} \min\{|x - y|_v, 1\} \leqslant \frac{1}{H_{\mathcal{O}(1)}(y)^{2+\varepsilon}}$$

.

For a proof, see ([**8**], Part D).

2.6. Schmidt subspace theorem

***Theorem 2.6.1* (Schmidt).** — *Let K be a number field. We fix an embedding of K into $\mathbf{C}$. $|\cdot|$ denotes the usual euclidean metric. Let $L_1, \cdots, L_n$ be n-linearly independent linear forms with coefficients in K in n variables. Then for any $\varepsilon > 0$, the primitive solutions to*

$$\prod_{i=1}^{n} |L_i(x)| < \frac{1}{H_{\mathcal{O}(1)}(x)^{\varepsilon}};\ x \in \mathbf{Z}^n$$

is contained in finitely many proper linear subspaces of $\mathbf{Q}^n$.

It is quite easy to see that Schmidt subspace theorem implies Roth's theorem. With the notations in (2.4.1), we want to show that the solutions to the equation

$$\left|z - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}};\ \frac{p}{q} \in \mathbf{Q} \qquad (*)$$

are finite. For $x = (p, q) \in \mathbf{Z}^2$, define $L_1(x) = q, L_2(x) = p - zq$. Then $(*)$ says

$$|L_1(x)L_2(x)| = |q(p - zq)| < \frac{1}{q^{\varepsilon}} \leqslant \frac{C}{\max\{|p|, |q|\}^{\varepsilon}}$$

where C is a constant depending on z. We conclude from Schmidt subspace theorem that such solutions lie in finitely many proper subspaces of $\mathbf{Q}^2$, which is the same thing as saying that solutions to $(*)$ are finite.

***Corollary 2.6.2* (Siegel-Thue-Schmidt).** — Let $(\xi_i)_{i=1}^n$ be real algebraic numbers such that $1, \xi_1, \cdots, \xi_n$ are linearly independent over $\mathbf{Q}$. Then for any $\varepsilon > 0$, there exists only finitely many rational n-tuples $(p_i/q)_{i=1}^n$ as solutions to

$$\left|\xi_i - \frac{p_i}{q}\right| < \frac{1}{q^{\frac{n+1}{n}+\varepsilon}};\ i = 1, \cdots, n.$$

The interesting thing here is the constant $\frac{n+1}{n}$. We shall define a new approximation constant β in chapter 4 and state in chapter 6 a generalization of this theorem.

CHAPTER 3

APPROXIMATING METRICS ON VARIETIES

Form now on let k be a number field, and let X be a projective variety over k (which is, in our situation, a geometrically integral regular projective scheme of finite fype over a field k of characteristic 0).

3.1. Archimedean case

For an archimedean place v of k, k_v is either $\mathbf{C}$ or $\mathbf{R}$, So $k_v \subset \mathbf{C}$ And v is just the place corresponding to the usual absolute value $|\cdot|$ on $\mathbf{C}$. Choose an embedding $i : X \to \mathbf{P}^n_k$. Then $X(\bar{k})$ is naturally a subset of $\mathbf{P}^n_k(\bar{k})$. We define, on $\mathbf{P}^n_{k_v}(\mathbf{C}) \times \mathbf{P}^n_{k_v}(\mathbf{C})$, a function as follows:

$$d_v(x, y) = \left(\frac{|\sum_{0 \leqslant i < j \leqslant n} x_i y_j - x_j y_i|^2}{(\sum_{i=0}^n |x_i|^2)(\sum_{i=0}^n |y_i|^2)} \right)^{[k_v:\mathbf{R}]/2}$$

where $x = [x_0 : \cdots : x_n]$ and $y = [y_0 : \cdots : y_n] \in \mathbf{P}^n_{k_v}(\mathbf{C})$ in terms of coordinates. we have $d_v(x, y) = 0 \iff [x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$.

Note that in general d does not satisfy triangle inequality, but since we only care about the behavior of it when approximating x, there is no need to make $X(\bar{k})$ into a usual metric space, as we shall see later.

3.2. Non-archimedean case

Now fix a non-archimedean place v, and an extension of v to $\bar{k}$. Choose an embedding $i : X \to \mathbf{P}^n_k$. We define the function $d_v : \mathbf{P}^n_k(\bar{k}) \times \mathbf{P}^n_k(\bar{k}) \to \mathbf{R}_{\geqslant 0}$ as follows:

$$d_v(x, y) = \frac{\max_{0 \leqslant i < j \leqslant n} |x_i y_j - x_j y_i|_v}{\max_{0 \leqslant i \leqslant n} |x_i|_v \max_{0 \leqslant j \leqslant n} |y_j|_v}$$

Geometric Interpretation: Given two different points x and y in $X(\bar{k})$, We choose a finite extension F of k such that $x, y \in X_F(F)$, We restrict v to F and denote by F_v the completion of F with respect to v and $\mathcal{O}_v$ the valuation ring of F_v. Let

$X_v = X \times_k F_v$ be the base change of X to F_v. Since $\mathbf{P}^n_{F_v}$ is the generic fiber of $\mathbf{P}^n_{\mathcal{O}_v}$, we can embed X_v in $\mathbf{P}^n_{\mathcal{O}_v}$ and take $\mathcal{X}_v$ to be its Zariski closure. That is, we have the following commutative diagram:

$$\begin{array}{ccc} X_v & \hookrightarrow & \mathcal{X}_v \\ \big\downarrow & & \big\downarrow \\ \mathbf{P}^n_{F_v} & \hookrightarrow & \mathbf{P}^n_{\mathcal{O}_v} \end{array}$$

$\mathcal{X}_v$ is called a *model* for X_v. Now x, y correspond to two sections $\delta_x, \delta_y : \mathrm{Spec}(F_v) \to X_v\mathbf{P}^n_{F_v}$, hence lift to two sections $\sigma_x, \sigma_y : \mathrm{Spec}(\mathcal{O}_v) \to \mathcal{X}_v \to \mathbf{P}^n_{\mathcal{O}_v}$ by the valuative criterion of properness. If we express x, y in coordinates: $x = [x_0, \cdots, x_n], y = [y_0, \cdots, y_n]$ with $x_i, y_i \in \mathcal{O}_v$ primitive, by which we mean that by multiplying suitable powers of a generator the maximal ideal such that all coordinates are in $\mathcal{O}_v$ and that at least one of their coordinates $\notin \mathfrak{m}_v$ (in other words, the gcd of their coordinates are units).

Let us first examine the image of $\sigma_x(\sigma_y)$ in $\mathbf{P}^n_{\mathcal{O}_v}$. The image of the morphism σ_x consists of two parts, namely the images of generic point and the special point:

$$\begin{array}{ccc} \mathrm{Spec}(F_v) & \xrightarrow{\delta_x(\delta_y)} & \mathbf{P}^n_{F_v} \\ \big\downarrow & & \big\downarrow f \\ \mathrm{Spec}(\mathcal{O}_v) & \xrightarrow{\sigma_x(\sigma_y)} & \mathbf{P}^n_{\mathcal{O}_v} \\ \big\uparrow & & \big\uparrow g \\ \mathrm{Spec}(\mathcal{O}_v/\mathfrak{m}_v) & \xrightarrow{g_x(g_y)} & \mathbf{P}^n_{\mathcal{O}_v/\mathfrak{m}_v} \end{array} \tag{3.2.1}$$

Assume for simplicity that x_0 is a unit in $\mathrm{Spec}(\mathcal{O}_v)$. Let η denote the generic point and $\mathfrak{m}$ the maximal ideal of $\mathrm{Spec}(\mathcal{O}_v)$. We have $\mathrm{Im}\,\delta_x \in \mathrm{D}_+(Z_0)$. The morphism f restricted to $\mathrm{D}_+(Z_0)$ corresponds to the morphism of rings

$$\tilde{f} : \mathcal{O}_v[Z_1/Z_0, \cdots, Z_n/Z_0] \to F_v[Z_1/Z_0, \cdots, Z_n/Z_0].$$

So

$$f(\delta_x(\eta)) = \sigma_x(\eta) = (Z_i/Z_0 - x_i/x_0)_{1 \leqslant i \leqslant n} \cap \mathcal{O}_v[Z_1/Z_0, \cdots, Z_n/Z_0].$$

Similarly, the morphism g restricted to $\mathrm{D}_+(Z_0)$ corresponds to the morphism of rings

$$\tilde{g} : \mathcal{O}_v[Z_1/Z_0, \cdots, Z_n/Z_0] \to \mathcal{O}_v/\mathfrak{m}_v[Z_1/Z_0, \cdots, Z_n/Z_0].$$

So

$$\begin{aligned} g(g_x(\mathfrak{m})) = \sigma_x(\eta) &= \tilde{g}^{-1}(Z_1/Z_0 - \overline{x_i/x_0})_{1 \leqslant i \leqslant n} \\ &= (Z_i/Z_0 - x_i/x_0)_{1 \leqslant i \leqslant n} + \mathfrak{m}_v[Z_1/Z_0, \cdots, Z_n/Z_0, 1]. \end{aligned}$$

After homogenization and since $x_0, \cdots, x_n$ are primitive, we see that

$$\sigma_x(\eta) = I_1 = (x_i Z_j - x_j Z_i)_{0 \leqslant i,j \leqslant n}.$$

and
$$\sigma_x(\mathfrak{m}) = I_2 = (x_i Z_j - x_j Z_i)_{0\leqslant i,j\leqslant n} + \mathfrak{m}_v[Z_0, \cdots, Z_n]$$
as homogeneous ideals of $\mathbf{P}^n_{\mathcal{O}_v}$. So in the principal open set $\mathrm{D}_+(Z_0) \simeq \mathbf{A}^n_{\mathcal{O}_v}$,
$$I_1 = (Z_i/Z_0 - x_i/x_0)_{1\leqslant i\leqslant n} \subsetneq (Z_i/Z_0 - x_i/x_0) + \mathfrak{m}_v[1, Z_i]_{1\leqslant i\leqslant n} = I_2$$
as prime ideals in $\mathrm{D}_+(Z_0)$. We see that the image of σ_x is in fact contained in $\mathrm{D}_+(Z_0)$.

Let $Z = \mathrm{Spec}(\mathcal{O}_v)\times_{\mathcal{X}_v}\mathrm{Spec}(\mathcal{O}_v) = \mathrm{Spec}(\mathcal{O}_v)\times_{\mathbf{P}^n_{\mathcal{O}_v}}\mathrm{Spec}(\mathcal{O}_v)$ be the fiber product of the two sections. Let I be the ideal of $\mathcal{O}_v$ generated by elements $\{x_i y_j - x_j y_i\}_{0\leqslant i,j\leqslant n}$. There exists a unique integer $p \geqslant 0$ such that $I = \mathfrak{m}_v^p$.We say that for $t \in \mathbf{N}$,
$$[x_0, \cdots, x_n] = [y_0, \cdots, y_n] \text{ modulo } \mathfrak{m}_v^t. \tag{$*$}$$
if $I \subset \mathfrak{m}_v^t$.

Proposition 3.2.1. — The following statements are equivalent:

1. Z is non-empty;
2. $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$ modulo $\mathfrak{m}_v$.

If any one of these statements is true then
($**$) $p \geqslant 1$ and the scheme Z is affine and is equal to $\mathrm{Spec}(\mathcal{O}_v/\mathfrak{m}^p)$.

Proof. — We have the following commutative diagram:
$$\begin{array}{ccccc} Z & \xrightarrow[\sigma_2]{} & \mathrm{Spec}(\mathcal{O}_v) & & \\ \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_y} & & \\ \mathrm{Spec}(\mathcal{O}_v) & \xrightarrow{\sigma_x} & \mathcal{X}_v & \longrightarrow & \mathbf{P}^n_{\mathcal{O}_v} \end{array} \tag{3.2.2}$$
($1 \Rightarrow 2$) Since $x \neq y$, the images of the generic point of $\mathrm{Spec}(\mathcal{O}_v)$ under σ_x and σ_y are different. So if $Z \neq \varnothing$, the commutative diagram (3.2.2) means that $\mathrm{Im}(\sigma_i) = \mathfrak{m}$ and that $\sigma_x(\mathfrak{m}) = \sigma_y(\mathfrak{m})$. Namely, as homogeneous ideal J in $\mathbf{P}^n_{\mathcal{O}_v}$,
$$J = (x_i Z_j - x_j Z_i)_{0\leqslant i,j\leqslant n} + \mathfrak{m}_v[Z_0, \cdots, Z_n] = (y_i Z_j - y_j Z_i)_{0\leqslant i,j\leqslant n} + \mathfrak{m}_v[Z_0, \cdots, Z_n].$$
We may assume it lies in the principal open set $\mathrm{D}_+(Z_0)$. Take into account the following relations:
$$Z_0(x_i y_j - x_j y_i) = y_i(x_0 Z_j - x_j Z_0) - (y_i Z_j - y_j Z_i) - y_j(x_0 Z_i - x_i Z_0) \in J;\ 0 \leqslant i, j \leqslant n.$$
By the assumption we get $x_i y_j - x_j y_i \in \mathfrak{m}_v$ for $0 \leqslant i, j \leqslant n$ hence $I \subset \mathfrak{m}_v$.
($2 \Rightarrow 1$) We shall prove ($**$). Then it follows directly that $Z \neq \varnothing$. We claim that $\exists i$ ($0 \leqslant i \leqslant n$) such that both x_i and y_i are units. Indeed, let $p \in \mathbf{N}$ such that $I = \mathfrak{m}^p$. Suppose we can find $i \neq j$ such that x_i, y_j are units but x_j, y_i are not. Then the element $x_i y_j - x_j y_i$ is a unit, which is a contradiction to the assumption of (2). From now on for simplicity we assume $x_0 = y_0 = 1$. Then the morphisms σ_x and σ_y in fact factorize through the principal open set $\mathrm{D}_+(Z_0) \simeq \mathbf{A}^n_{\mathcal{O}_v}$:
$$\mathrm{Spec}(\mathcal{O}_v) \xrightarrow{\sigma_x(\sigma_y)} \mathbf{A}^n_{\mathcal{O}_v} \longrightarrow \mathbf{P}^n_{\mathcal{O}_v}.$$
So
$$Z = \mathrm{Spec}(\mathcal{O}_v) \times_{\mathbf{A}^n_{\mathcal{O}_v}} \mathrm{Spec}(\mathcal{O}_v) = \mathrm{Spec}(\mathcal{O}_v \otimes_{\mathcal{O}[Z_1,\cdots,Z_n]} \mathcal{O}_v).$$

To determine the $\mathcal{O}_v$-algebra $\mathcal{O}_v \otimes_{\mathcal{O}[Z_1,\cdots,Z_n]} \mathcal{O}_v$, first notice that the natural map $\varphi : \mathcal{O}_v \to \mathcal{O}_v \otimes_{\mathcal{O}[Z_1,\cdots,Z_n]} \mathcal{O}_v$ is surjective (because any element of the form $a \otimes b = ab(1 \otimes 1)$ is the image of ab in $\mathcal{O}_v$). It suffices to determine the kernel of φ. We easily see that the kernel I' is generated by elements $\{x_i - y_i\}_{1 \leqslant i \leqslant n}$ (because $x_i \otimes 1 = 1 \otimes y_i$ in $\mathcal{O}_v \otimes_{\mathcal{O}[Z_1,\cdots,Z_n]} \mathcal{O}_v$). Obviously $I' \subset I$. On the other hand we have relations $x_i y_j - x_j y_i = y_j(x_i - y_i) - y_i(x_j - y_j)$ for all $i, j \geqslant 1$ so $I \subset I'$ hence $I = I'$ and $\mathcal{O}_v / I \cong \mathcal{O}_v \otimes_{\mathcal{O}[Z_1,\cdots,Z_n]} \mathcal{O}_v$. So $Z = \operatorname{Spec}(\mathcal{O}_v / \mathfrak{m}^p)$. The proof is complete. □

We now define

$$d'_v(x,y) = \begin{cases} 1, & \text{if } Z = \varnothing \\ 1/(\operatorname{Card}\Gamma(Z, \mathcal{O}_Z)), & \text{otherwise} \end{cases}$$

We now show the following proposition, which gives a geometric meaning to non-archimedean distance functions.

Proposition 3.2.2. — $d'_v(x,y) = d_v(x,y)$.

Proof. — Let q be the prime number such that v is the extension of q-adic absolute value $|\cdot|_q$ on $\mathbf{Q}$ ($|a|_q = q^{-\operatorname{ord}_q(a)}$). Let $d_q = [F_v : \mathbf{Q}_q]$, $f_q = [\mathcal{O}_v/\mathfrak{m} : \mathbf{F}_q]$ (residue degree), e_q be the ramification index ($q\mathcal{O}_v = \mathfrak{m}^{e_q}$). Let π be a generator for the maximal ideal $\mathfrak{m}$ such that $q = \pi^{e_q} u$ where u is a unit in $\mathcal{O}_v$. Then we have

$$|\pi|_v = |\operatorname{Norm}_{F_v/\mathbf{Q}_q}(\pi)|^{\frac{1}{d_q}} = |q^{-\frac{d_q}{e_q}}|_q = q^{-f_q}.$$

So

$$|\pi|_v^p = q^{-pf_q} = 1/(\operatorname{Card}(\mathcal{O}_v/\mathfrak{m}^p)) = 1/(\operatorname{Card}\Gamma(Z, \mathcal{O}_Z)).$$

On the other hand, according to the way we normalize the coordinates of x and y, we have that

$$\max_{0 \leqslant i \leqslant n} |x_i|_v = \max_{0 \leqslant j \leqslant n} |y_j|_v = 1.$$

By the previous proposition, the ideal of $\mathcal{O}_v$ generated by the elements $\{x_i y_j - x_j y_i\}_{0 \leqslant i,j \leqslant n}$ is either $\mathfrak{m}^p$ (if $Z \neq \varnothing$) or equal to the whole ring (if $Z = \varnothing$). Combine the formulas we have obtained so far, we get

$$d_v(x,y) = \max_{0 \leqslant i,j \leqslant n} |x_i y_j - x_j y_i| = \left\{ \begin{array}{ll} 1, & \text{if } Z = \varnothing \\ |\pi|_v^p, & \text{otherwise} \end{array} \right\} = d'_v(x,y).$$

□

We remark here that it is easy to show that the distance function defined over is invariant under finite field extension.

3.3. Equivalence of distance functions

In the definition of distance function we fix an embedding of X into a projective space. The value of the distance function depends thus on the embedding. We shall see that the results appearing later on only depend on the equivalence class of distance functions. And we have

Proposition 3.3.1. — Let d_v and d'_v be two distance functions coming from two different projective embeddings of X. Then for any finite field extension K/k, d_v and d'_v are equivalent on $X(K_v) \times X(K_v)$.

And we also the following result for doing local computations when considering approximation by a sequence of points.

Proposition 3.3.2. — Let x be a point of $X(\bar{k})$ and K be a finite extension of k which is also a field of definition for x. Let U be any open affine open neighborhood of x in X_K and $Z_1, \cdots, Z_r$ be regular functions on U that generate the maximal ideal defining x. Then for any sequence (x_i) of points of $U(K_v)$ such that $d_v(x_i, x) \to 0$, the functions $d_v(x, \cdot)$ and $\max(\|Z_1(\cdot)\|, \cdots, \|Z_r(\cdot)\|)$ are equivalent on the set (x_i).

For proofs of these results, we refer to ([**10**] 2.13-2.17).

CHAPTER 4

APPROXIMATION CONSTANTS α AND β

Throughout this section, X denotes an irreducible smooth projective scheme of finite type over a number field k. Let $\mathcal{M}_k$ denote the set of places of k, L be a **Q**-line bundle. Fix a place $v \in \mathcal{M}_k$. Given $x \in X(k)$, we are interested in approximating the algebraic point x by rational points.

4.1. The approximation constant α

Definition 4.1.1. — A line bundle L is said to have *Northcott property* if for any constant B, the set of points having *bounded height* B

$$\{x \in X(k) : \ H_L(x) \leqslant B\}$$

is finite.

***Theorem 4.1.2* (Northcott).** — *For any integers n, d and real number B, there are at most finitely many points of $\mathbf{P}^n_{\mathbf{Q}}(\bar{\mathbf{Q}})$ having height $\leqslant B$ and degree $\leqslant d$.*

For a proof of this theorem, see ([**12**], §2.4).

Corollary 4.1.3. — Any ample line bundle on X satisfies the Northcott property.

Definition 4.1.4. — Let $(x_i) \in X(k)$ be a sequence of pairwise distinct points such that $d_v(x, x_i) \to 0$. We define the sets

$$A^{v,L}_x((x_i)) = \{\gamma \in \mathbf{R} : \exists C > 0 \,\text{such that}\, \forall i, \ d_v(x_i, x)^\gamma H_L(x_i) \leqslant C\}$$

and

$$A^{v,L}_x = \bigcup_{(x_i) \in X(k), \ d_v(x_i, x) \to 0} A^{v,L}_x((x_i)).$$

Here the height function is taken to be any one in the equivalence class or quasi-equivalence class of H_L. If there is no sequence (x_i) such that $d_v(x_i, x) \to 0$, we set $A^{v,L}_x = \varnothing$.

It is clear that if the set $A^{v,L}_x((x_i))$ is nonempty, then it is an interval unbounded to the right.

Definition 4.1.5. — We define $\alpha_x^{v,L}((x_i))$, the approximation constant with respect to the sequence (x_i), to be the infimum of the set $A_x^{v,L}((x_i))$. We define the *approximation constant* $\alpha_x^v(L)$ to be the infimum of the set $A_x^{v,L}$. That is, it is the infimum of all approximation constants of sequences of points in $X(k)$ converging to x (in the sense of place v).

We explain some examples for the extreme cases of $\alpha_x^v(L) = \infty$ and $\alpha_x^v(L) = -\infty$. For the first case Take X to be a curve of genius $\geqslant 2$ defined over k. Then by a famous theorem due to Faltings, Card $X(k)$ is finite. So for any line bundle L on X and any place $v \in \mathcal{M}_k$, we have $A_x^{v,L} = \varnothing$. For the second case , take $X = \mathbf{P}^1_{\mathbf{Q}}$, $L = \mathcal{O}(-1)$ and v to be the finite place corresponding to a prime number p. Fix an positive integer n. Choose a prime number $q > p^{n-1}$. We define sequence (x_i) as $x_i = [1 : (pq)^i]$. This sequence tends to the point $x = [1 : 0]$ in the v-adic distance. But we have

$$d_v(x_i, x)^{-n} H_{\mathcal{O}(-1)}(x_i) = p^{ni}\Big(\prod_{w\in\mathcal{M}_{\mathbf{Q}}} \max(|1|_w, |pq|_w^i)\Big)^{-1} = \left(\frac{p^{n-1}}{q}\right)^i.$$

So $-n \in A_x^{v,\mathcal{O}(-1)}$. Hence $\alpha_x^v(\mathcal{O}(-1)) = -\infty$.

Proposition 4.1.6. — Let $x \in X(\bar{k})$ and let L be the line bundle which satisfies the Northcott property. Then $\alpha_x^v(L) \geqslant 0$ and is equal to

$$b_x(L) = \sup\{\nu \geqslant 0 : \forall C > 0,\ \text{the set } \{y \in X(k) : d_v(x,y)^\nu H_L(y) < C\} \text{ is finite}\}$$

Proof. — We assume that $\alpha_x^v(L) < +\infty$. The case $\alpha_x^v(L) = +\infty$ proceeds in the same way. Suppose $\alpha_x^v(L) < 0$. Then there exists $\lambda < 0$ and a sequence of pairwise different points $(x_i) \to x$ in $X(k)$ such that $\{d_v(x_i, x)^\nu H_L(x_i)\}$ is bounded from above. But $\{H_L(x_i)\}$ is unbounded and $d_v \leqslant 1$. This causes a contradiction. So $\alpha_x^v(L) \geqslant 0$. It is easy to see that $\alpha_x^v(L) \geqslant b_x(L)$. We need to show the inverse inequality. We may assume $\alpha_x^v(L) > 0$ (otherwise it is trivial). Fix $\delta \in]0, \alpha_x^v(L)[$. Suppose $\exists C > 0$ such that the set $\{y \in X(k) : d_v(x,y)^\delta H_L(y) < C\}$ is infinite. Then we can extract a sequence of pairwise different points $(x_i) \to x$, such that $\{d_v(x_i, x)^\delta H_L(x_i)\}$ is bounded by C. This contradicts to the definition of $\alpha_x^v(L)$. So $b_x(L) \geqslant \delta$ and hence $b_x(L) \geqslant \alpha_x^v(L)$. □

Remark 4.1.7. — At this point it is clear that, combining Northcott's theorem, the approximation constant α is indeed a generalization of the "approximation exponent" μ defined in Chap. 2 in the particular case $X = \mathbf{P}^1_{\mathbf{Q}}$. And we have the relation

$$\alpha_z^\infty(\mathcal{O}_{\mathbf{P}^1_{\mathbf{Q}}}(1)) = \mu(z)^{-1} \text{ for all } z \in \mathbf{R}.$$

In particular Liouville's theorem can be written as

$$\alpha_z^\infty(\mathcal{O}_{\mathbf{P}^1_{\mathbf{Q}}}(1)) \geqslant \frac{1}{d}$$

for all $z \in \mathbf{R}$ of degree d. And Roth's theorem can also be stated as

$$\alpha_x^\infty(\mathcal{O}_{\mathbf{P}^1_{\mathbf{Q}}}(1)) = \frac{1}{2}$$

for all irrational algebraic number x. This allows us to examine approximation of rational points on arbitrary varieties.

We list several properties of the approximation constant α below.

Properties of the approximation constant α.

1. For ample line bundle L, the constants $\alpha_x^{v,L}((x_i))$ and $\alpha_x^v(L)$ only depend on the equivalence class of the line bundle L in $N^1(X) \otimes \mathbf{Q}$ (Proposition 1.4.6).
2. The approximation constant is determined locally for the v-adic topology. (Since the distance function can be computed through the local equations of any open neighborhood of x (Proposition 3.3.2)).
3. If L is a $\mathbf{Q}$-effective line bundle and $x \in X \smallsetminus Ba(L)(\bar{k})$, then by Proposition 1.4.4 there exists $C > 0$ such that $H_L \geqslant C$. For any sequence $(x_i) \subset X \smallsetminus Ba(L)(k)$ such that $d_v(x_i, x) \to 0$, we conclude that $\alpha_x^{v,L}((x_i)) \geqslant 0$, hence $\alpha_x^v(L) \geqslant 0$.

One may note that the definition of α is slightly different from the "approximation exponent" appearing in Chap. 2. Namely, the exponent is moved to the distance function. It is natural to do so because comparing the following proposition with (1.3.6), one may discover that it has astonishing similarities to the Seshadri constant.

Proposition 4.1.8. — The following statements are true.

1. For any $x \in \mathbf{P}^n(k)$, $\alpha_x^v(\mathcal{O}(1)) = 1$.
2. Let $x \in X(\bar{k})$ and L be ample, then $\alpha_x^v(L) > 0$.
3. $\alpha_x^v(\cdot)$ is a concave function on $N^1(X)_{\mathbf{Q}}$. More precisely, for any $\mathbf{Q}$-line bundles H_1, H_2 (except for the case that $\{\alpha_x^v(H_1), \alpha_x^v(H_2)\} = \{-\infty, +\infty\}$), we have
$$\alpha_x^v\left(aH_1 + bH_2)\right) \geqslant a\alpha_x^v(H_1) + b\alpha_x^v(H_2).$$
4. Let L_1 and L_2 be asymptotically base point free line bundles on X_1 and X_2 and $x_1 \in X_1(\bar{k}), x_2 \in X_2(\bar{k})$.
   If neither x_1 nor x_2 are defined over k, then
$$\alpha_{x_1,x_2}^v(L_1 \boxtimes L_2) \geqslant \alpha_{x_1}^v(L_1) + \alpha_{x_2}^v(L_2).$$
   If x_1 is over k but x_2 is not, then
$$\alpha_{x_1,x_2}^v(L_1 \boxtimes L_2) = \alpha_{x_2}^v(L_2).$$
   If x_1, x_2 are both defined over k, then
$$\alpha_{x_1,x_2}^v(L_1 \boxtimes L_2) = \min\{\alpha_{x_1}^v(L_1), \alpha_{x_2}^v(L_2)\}.$$

Statement (2) follows immediately from (1). (3) and (4) are straightforward from definition. For details we refer to [**10**] (2.14) and (2.15). We only prove (1).

Proof. — We may assume $x = [1 : 0 : \cdots : 0]$. We fix a sequence of k-points (x_i) such that $d_v(x_i, x) \to 0$. By Proposition 3.3.2, we know that locally the distance function is equivalent to $\max_{1 \leqslant i \leqslant n} \|Z_i/Z_0\|_v$. By passing to a subsequence of (x_i), we may assume that $\max_{1\leqslant i\leqslant n}\|Z_i(x_i)/Z_0(x_i)\|_v<1$ and that $\max_{1\leqslant i\leqslant n}\|Z_i(x_i)\|_v=\|Z_1(x_i)\|_v$. Then for any $\lambda>0$, using Proposition 3.3.2 and the product formula,

$$\begin{aligned}
d_v(x_i,x)^\lambda H_{\mathcal{O}(1)}(x_i) &= \frac{\|Z_1(x_i)\|_v^\lambda}{\|Z_0(x_i)\|_v^\lambda}\prod_{w\in\mathcal{M}_k}\max_{0\leqslant i\leqslant n}(\|Z_i(x_i)\|_w)\\
&= \frac{\|Z_1(x_i)\|_v^\lambda}{\|Z_0(x_i)\|_v^{\lambda-1}}\prod_{w\in\mathcal{M}_k,w\neq v}\max_{0\leqslant i\leqslant n}(\|Z_i(x_i)\|_w)\\
&\geqslant \frac{\|Z_1(x_i)\|_v^\lambda}{\|Z_0(x_i)\|_v^{\lambda-1}}\prod_{w\in\mathcal{M}_k,w\neq v}\|Z_1(x_i)\|_w\\
&= \frac{\|Z_1(x_i)\|_v^{\lambda-1}}{\|Z_0(x_i)\|_v^{\lambda-1}}.
\end{aligned}$$

As $\frac{\|Z_1(x_i)\|_v}{\|Z_0(x_i)\|_v}\to 0$, in order to have it to be bounded we must have $\lambda\geqslant 1$. This proves that $\alpha_x^v(\mathcal{O}(1))\geqslant 1$.

We next show the equality, namely for any v we can always find some sequence (x_i) tending to x with $\alpha_x^{v,\mathcal{O}(1)}((x_i))=1$. Since we can approximate x through rational lines in $\mathbf{P}^n$ it suffices to treat the case of $\mathbf{P}^1$. First suppose v is archimedean, we claim that there exists $D>1$ such that the cardinal of the set

$$T=\{y\in\mathcal{O}_k:|y|_w\leqslant D \text{ for all archimedean } w\neq v\}$$

is infinite. In fact, we embed $\mathcal{O}_k$ into the Euclidean space $E=\prod_{w|\infty}k_w$. Its image in E is a lattice. We take $D>1$ such that the rectangle $\prod_{w|\infty}\{|x_w|\leqslant D\}$ in E contains a fundamental domain of the lattice. Then the set T can be seen as a cylinder in E containing infinitely many lattice points. We choose a sequence (y_i) in T such that $|y_i|_v\to\infty$ and define the sequence of k-points (x_i) as $x_i=[y_i:1]$. Then $d_v(x_i,x)\to 0$ and

$$d_v(x_i,x)H_{\mathcal{O}(1)}(x_i)=|y_i|_v^{-1}\prod_w\max(1,|y_i|_w)\leqslant\prod_{w|\infty,\neq v}\max(1,|y_i|_w)\leqslant D^r$$

where r is the dimension of E as an $\mathbf{R}$-vector space. Therefore $\alpha_x^{v,\mathcal{O}(1)}((x_i))\leqslant 1$. Now suppose v is non-archimedean. Let p be the prime number corresponding to the finite place v. We define the sequence of k-points (x_i) as $x_i=[1:p^i]$. Then $d_v(x_i,x)\to 0$. And we have

$$\begin{aligned}
d_v(x_i,x)H_{\mathcal{O}(1)}(x_i) &= |b|^i\prod_{w\in\mathcal{M}_k}\max(1,|b^i|_w)\\
&= |b|_v^i\prod_{w|\infty}|b|_w^i\\
&= |b|_v^i\prod_{w|\infty}|b|_w^i\prod_{u\text{ finite},\neq v}|b|_u^i\\
&= 1
\end{aligned}$$

by the product formula. Therefore we get $\alpha_x^{v,\mathcal{O}(1)}((x_i))\leqslant 1$. □

4.2. The approximation constant β

Let L be an ample line bundle on X. Let $x \in X(\bar{k})$. Let K be a field of definition of x and $\pi : \widetilde{X_K} \to X_K$ be the blow up of X_K along x with exceptional divisor E. For $\gamma \geqslant 0$, let $L_\gamma = \pi^* L - \gamma E$.

Definition 4.2.1. — We define the *density* function of the pair (L, x) to be

$$f(\gamma) = \frac{\operatorname{Vol}(L_\gamma)}{\operatorname{Vol}(L)} \ (\gamma \in \ [0, +\infty[).$$

The *approximation constant* β is defined as

$$\beta(L, x) = \int_0^\infty f(\gamma) d\gamma.$$

Since base changes of field extensions are flat, the value of $\beta(L, x)$ is invariant under field extension. Namely it can be computed over any field of definition of x.

In the first chapter we have seen the Seshadri constant is the largest number ε such that L_ε lies in the nef cone of $\widetilde{X}$. We let γ_e be the supremum of positive real numbers γ such that L_γ lies in the effective cone of $\widetilde{X}$. By Kleiman's theorem, we see that γ is also the supremum of positive real numbers γ such that L_γ lies in the big cone of $\widetilde{X}$ and that ε is less than γ_e. Also from Proposition 1.1.4 we see that $\operatorname{Vol}(L_.)$ is a decreasing function hence the same holds for the density function. Furthermore, since the volume function is continuous, it suffices to consider the case when γ is rational. Suppose that $\gamma \in \mathbf{Q}$ and L_γ is big. By multiplying some integer, we also use the same notation to denote the equivalent class of divisors corresponding to L_γ. We fix a very ample line bundle H of $\widetilde{X}$. Then

$$0 < \deg(L_\gamma) = H^{n-1} \cdot L_\gamma = \pi^* L \cdot H^{n-1} - \gamma E \cdot H^{n-1} = \pi^* L \cdot H^{n-1} - \gamma \deg E.$$

So

$$\gamma < \frac{\pi^* L \cdot H^{n-1}}{\deg E}.$$

Hence

$$\gamma_e \leqslant \frac{\pi^* L \cdot H^{n-1}}{\deg E}.$$

Since $\operatorname{Vol}(L_\gamma) > 0 \Leftrightarrow L_\gamma$ is big,

$$\beta(L, x) = \int_0^{\gamma_e} f(\gamma) d\gamma$$

is a well-defined non-negative number.

Example 4.2.2. — We now compute the β and γ_e of the very ample line bundle $L = \mathcal{O}(1)$ on the projective space $\mathbf{P}^n_k$. We may assume $x = [1 : 0 : \cdots : 0]$. For $\gamma \in \mathbf{Q}_{\geqslant 0}$, let $m \in \mathbf{N}$ such that $m\gamma$ is an integer. In the view of (1.1.4), the space of global sections $H^0(\widetilde{\mathbf{P}^n_k}, mL_\gamma)$ can be identified with the space of homogeneous polynomials of degree m in $n+1$ variables which vanish to order $\geqslant m\gamma$ at point x. Then

$$\mathrm{Card}(H^0(\widetilde{\mathbf{P}^n_k}, mL_\gamma)) = \binom{m+n}{n} - \binom{m\gamma+n}{n} = \frac{m^n(1-\gamma^n)}{n!} + O(m^{n-1}).$$

So $\mathrm{Vol}(mL_\gamma) = m^n \mathrm{Vol}(L_\gamma) > 0 \Leftrightarrow \gamma < 1$. This implies that $\gamma_e = 1 = \varepsilon(L,x)$, which means that L_γ is nef $\Leftrightarrow$ L_γ is pseudoeffective. So $\mathrm{Vol}(L_\gamma) = 1-\gamma^n$ and the density function is given by $f(\gamma) = 1-\gamma^n$ when $\gamma < 1$. Hence

$$\beta(L,x) = \int_0^{\gamma_e} f(\gamma)d\gamma = \frac{n}{n+1}.$$

Next we state an important estimation of the volume function. For any ample line bundle L, we know that when $\gamma < \varepsilon(L,x)$, $\mathrm{Vol}(L_\gamma)$ can be computed directly through self-intersection number thanks to the asymptotic Riemann-Roch formula. However when $\gamma \in]\varepsilon(L,x), \gamma_e]$, L_γ is not nef. Nevertheless, we have the following lower bound for the volume function. The inequality can be strict. See also the remark after Theorem 4.2.4.)

Proposition 4.2.3. — For any $\gamma \geqslant 0$, $\mathrm{Vol}(L_\gamma) \geqslant \mathrm{Vol}(L) - \mathrm{mult}_x X \cdot \gamma^n$

Proof. — We may assume γ is rational. We fix $m \in \mathbf{N}$ such that $m\gamma$ is an integer. For any integer l, we denote by lY the non-reduced closed subscheme of $\widetilde{X}$ defined by the coherent sheaf of ideals $\mathfrak{I}_E^l$. We have the exact sequence

$$0 \to mL_\gamma \to mL_0 \to mL_0 \mid_{m\gamma Y} \to 0.$$

Hence

$$0 \to H^0(\widetilde{X}, mL_\gamma) \to H^0(\widetilde{X}, mL_0) \to H^0(m\gamma Y, mL_0 \mid_{m\gamma Y}).$$

Since $H^0(\widetilde{X}, mL_0) = H^0(X, mL)$ and L is ample, we see that by asymptotic Riemann-Roch,

$$h^0(mL_0) = \frac{\mathrm{Vol}(L)}{n!} m^n + O(m^{n-1}).$$

We now estimate $H^0(Y, mL_0 \mid_Y)$. Since L_0 restricts to the trivial line bundle on E, we have $mL_0 \mid_{m\gamma Y} \cong \mathcal{O}_{m\gamma Y}$. For any integer l the exact sequence of sheaves

$$0 \to \mathcal{O}_Y(-lY) \to \mathcal{O}_{(l+1)Y} \to \mathcal{O}_{lY} \to 0$$

induces the estimation

$$h^0(\mathcal{O}_{(l+1)Y}) \leqslant h^0(\mathcal{O}_Y(-lY)) + h^0(\mathcal{O}_{lY}).$$

Inductively we get that

$$h^0(\mathcal{O}_{m\gamma Y}) \leqslant \sum_{l=0}^{m\gamma-1} h^0(\mathcal{O}_Y(-lY)).$$

Now we fix any embedding $X \hookrightarrow \mathbf{P}^n_k$ so that Y is a closed subvariety of $\mathbf{P}^{n-1}_k$. We see from Proposition 1.1.2 that $\mathcal{O}_Y(-lY) = \mathcal{O}_Y(l) = \mathcal{O}_{\mathbf{P}^{n-1}_k}(l) \mid_Y$. Note that for $l \gg 0$,

$h^0(\mathcal{O}_Y(-lY))$ is equal to the Hilbert polynomial of the variety Y in $\mathbf{P}_k^{n-1}$. That is

$$h^0(\mathcal{O}_Y(-lY)) = \frac{\operatorname{mult}_x X}{(n-1)!} l^{n-1} + O(l^{n-2}) \text{ for } l \gg 0.$$

Since $\sum_{l=0}^{t} l^{n-1} = \frac{t^n}{n+1} + O(t^{n-1})$, we get that

$$H^0(Y, mL_0 \mid_Y) \leqslant \frac{\operatorname{mult}_x X}{n!}(m\gamma)^n + O(m^{n-1}).$$

By the definition of volume we get the desired inequality. □

Theorem 4.2.4. — *With notation as above,*

$$\beta(L,x) \geqslant \frac{n}{n+1}\sqrt[n]{\frac{\operatorname{Vol}(L)}{\operatorname{mult}_x X}} \geqslant \frac{n}{n+1}\varepsilon(L,x) \tag{4.2.4.1}$$

Proof. — The second inequality follows directly from Corollary 1.3.5.
Set

$$g(\gamma) = 1 - \frac{\operatorname{mult}_x X}{\operatorname{Vol}(L)}\gamma^n.$$

Then $\varpi = \sqrt[n]{\frac{\operatorname{Vol}(L)}{\operatorname{mult}_x X}}$ is the real root of g. From the proposition above we know that $g \leqslant f$ where f is the density function. Hence $\varpi \leqslant \gamma_e$. We get that

$$\beta(L,x) = \int_0^{\gamma_e} f(\gamma)d\gamma \geqslant \int_0^{\varpi} g(\gamma)d\gamma = \frac{n}{n+1}\sqrt[n]{\frac{\operatorname{Vol}(L)}{\operatorname{mult}_x X}}.$$

□

Remark 4.2.5. — The inequalities above can be strict. This is the case for $\mathbf{P}^1 \times \mathbf{P}^1$.

The following is some elementary properties of the approximation constant β, the proof of which is straightforward from the definition and the theorem above.

Proposition 4.2.6. — Let L be an ample line bundle on X, and $x \in X(k)$. Then the following two statements are true:

1. $\beta(L,x) > 0$;
2. For any $a \in \mathbf{Q}_{\geqslant 0}$, $\beta(aL,x) = a\beta(L,x)$.

A probabilistic interpretation of the constant β. In the spirit of Proposition 1.1.4, the space of global sections of L_γ can be viewed as the subspace of the space of global sections of L consists of sections which vanish at order $\geqslant \gamma$ at point $x \in X(k)$. We observe that $f(0) = 1, f(\gamma_e) = 0$. By ([**1**], Theorem A), the volume function is first differentiable. So we can think of the function $g(\gamma) = 1 - f(\gamma)$ as the probability of the event of sections vanishing to order $\leqslant \gamma$ at x with probability measure $g'(\gamma) = -f'(\gamma)$ on $\mathbf{R}_{\geqslant 0}$. We now compute the expectation

$$E(L,x) = \int_{\mathbf{R}_{\geqslant 0}} \gamma g'(\gamma)d\gamma = -\int_0^{\gamma_e} \gamma f'(\gamma)d\gamma = \gamma f(\gamma)\mid_{\gamma=0}^{\gamma=\gamma_e} + \int_0^{\gamma_e} f(\gamma)d\gamma = \beta(L,x).$$

So we can interpret β as the expected order of vanishing of sections at point x.

We end up this section with some conclusions on the possibility of achieving equality in (4.2.4.1).

Definition 4.2.7. — Let L be an $\mathbf{Q}$-ample line bundle and $x \in X(\bar{k})$. We say that the variety X is *Seshadri exceptional* with respect to L and x if

$$\varepsilon(L, x) = \left(\frac{(L^{\dim X})}{\operatorname{mult}_x X}\right)^{1/\dim X}.$$

Namely, the second inequality in (4.2.4.1) is an equality.

Theorem 4.2.8. — *With the above notation, the following conditions are equivalent:*

1. $\beta(L, x) = \frac{n}{n+1}\sqrt[n]{\frac{\operatorname{Vol}(L)}{\operatorname{mult}_x X}}$;
2. *X is Seshadri exceptional with respect to L and x;*
3. $\varepsilon(L, x) = \gamma_e$.

In other words, if any of the "$\geqslant$" in (4.2.4.1) is "$=$", then the other one must be. For a proof of this result we refer to ([**10**], Theorem 9.1).

4.3. An example: $\mathbf{P}^1 \times \mathbf{P}^1$

As an illuminating example, we shall compute and compare the approximation constants and Seshadri constant for any k-point of the variety $X = \mathbf{P}^1_k \times \mathbf{P}^1_k$ and the line bundle $L = \mathcal{O}(a, b)$ $(a, b \in \mathbf{N}_{\geqslant 1}, a \leqslant b)$. From Propositions 1.3.6 and 4.1.8, we obtain that $\varepsilon(L, x) = \alpha^v_x(L) = \min(a, b) = a$.

Next we compute β. We may assume the k-point x is $[1:0] \times [1:0]$. We choose the affine open set $U_0 \times U_0 \cong \mathbf{A}^2_k$ containing x. The space of global sections of L can be identified with products of homogeneous polynomials of degree a and b in two variables each, which is the same as, when restricted to $\mathbf{A}^2_k$, the space W of polynomials in two variables z_1, z_2 spanned by the monomials of the form $z_1^{c_1} z_2^{c_2}$ with $0 \leqslant c_1 \leqslant a, 0 \leqslant c_2 \leqslant b$. So $h^0(X, L) = \dim_k(W) = ab$. Since L is ample and X is smooth, $\operatorname{Vol}(L) = \dim X \cdot h^0(X, L) = 2ab$. For any $m \in \mathbf{N}$, denote by W_m the space of global sections of mL. Let $\pi : \widetilde{X} \to X$ be the blow up of X along x with exceptional divisor E. Fix $\gamma \in \mathbf{Q}_{>0}$ and $m \in \mathbf{N}_{>0}$ such that $m\gamma \in \mathbf{N}$. the space of global sections of the line bundle $mL_\gamma = m\pi^*(L) - m\gamma E$ can be identified with products of homogeneous polynomials of degree a and b in two variables each that vanish to order $\geqslant m\gamma$ at x, which is the same as the subspace $W_{m,\gamma}$ of W_m spanned by the monomials of the form $z_1^{c_1} z_2^{c_2}$ with $c_1 + c_2 \geqslant m\gamma$. So we have $h^0(\widetilde{X}, mL_\gamma) > 0 \Leftrightarrow \gamma < a + b$. Thus $\gamma_e = a + b > \varepsilon(L, x)$. In other words, the set of bases for the subspace $W_{m,\gamma}$ can be viewed as the lattice points lying in the polygon $P_{m,\gamma}$ in $\mathbf{R}^2$ defined in coordinates (x, y) as follows:

$$P_{m,\gamma} = \left\{ (x, y) \in \mathbf{R}^2 : \begin{array}{c} 0 \leqslant x \leqslant ma; \\ 0 \leqslant y \leqslant mb; \\ x + y \geqslant m\gamma \end{array} \right\}.$$

By the definition of volume, $\mathrm{Vol}(L_\gamma)$ is computed as the limit of number of lattice points in $P_{m\gamma}$ as $m \to \infty$. This limit tends the area of the polygon P_γ

$$P_\gamma = \left\{ (x,y) \in \mathbf{R}^2 : \begin{array}{l} 0 \leqslant x \leqslant a; \\ 0 \leqslant y \leqslant b; \\ x + y \geqslant \gamma \end{array} \right\}.$$

By computation we get the density function f as follows:

$$f(\gamma) = \frac{\mathrm{Vol}(L_\gamma)}{\mathrm{Vol}(L)} = \begin{cases} 1 - \frac{\gamma^2}{2ab} & \gamma \in [0,a]; \\ 1 + \frac{a}{2b} - \frac{\gamma}{b} & \gamma \in [a,b]; \\ \frac{(a+b-\gamma)^2}{2ab} & \gamma \in [b, a+b]. \end{cases}$$

whereas the function $g(\gamma) = 1 - \frac{\mathrm{mult}_x X}{\mathrm{Vol}(L)} \gamma^n = 1 - \frac{\gamma^2}{2ab}$ decreases strictly faster than f when $\gamma \in]a, a+b]$. This also confirms that the asymptotic Riemann-Roch formula would be false if L is not nef. We conclude that $\beta(L,x) = \int_0^{a+b} f(\gamma) d\gamma = \frac{a+b}{2}$. The inequality in Theorem 4.2.4 is just

$$\frac{a+b}{2} > \frac{\sqrt{8ab}}{3} > \frac{2a}{3}.$$

CHAPTER 5

STABILITY AND FALTINGS-WÜSTHOLZ THEOREM

5.1. Stability

Definition 5.1.1. — Let K be a field of characteristic 0 and V be a finite dimensional vector space over K. For a sequence of strictly increasing real numbers

$$0 \leqslant p_0 < p_1 < \cdots < p_m < p_{m+1}$$

and a filtration $F^p V$ of subspaces

$$0 = F^{p_{m+1}} V \subset F^{p_m} V \subset \cdots \subset F^{p_1} V \subset F^{p_0} V = V$$

We define the *slope* $\mu(V)$ of this filtration by

$$\mu(V) = \frac{\mathrm{wt}(V)}{\dim(V)} = \frac{1}{\dim V} \sum_{i=0}^{m} p_i \dim(F^{p_i}/F^{p_{i+1}}).$$

Note that for every non-zero subspace $W \subsetneq V$ (resp. quotient space $V \twoheadrightarrow M = V/W$), the filtration $F^p V$ naturally induces a filtration $F^p W$ on W (resp. $F^p M$ on M) by taking $F^{p_j} W = F^{p_j} V \cap W$ (resp. $F^{p_j} M = F^{p_j} V/(F^{p_j} V \cap W)$). From now on, when speaking of a non-zero subspace or a non-zero quotient space, we always take its induced filtration.

Since there is a non-canonical isomorphism $V \cong \oplus_i F^{p_i} V/F^{p_{i+1}} V$, we see that the slope of V can be viewed as associating a weight (p_i) on each part $F^{p_i} V/F^{p_{i+1}} V$, then calculating the average of them.

Given two filtered vector spaces $(V_1, F^p V_1)$, $(V_2, F^q V_2)$, we can also define a filtration over $V = V_1 \otimes V_2$. Suppose $F^p V_1$ (resp. $F^q V_2$) is given by $\{p_i\}_{i=0}^{m+1}$ and $\{F^{p_i} V_1\}_{i=0}^{m+1}$ (resp.$\{q_j\}_{j=0}^{n+1}$ and $\{F^{q_j} V_2\}_{j=0}^{n+1}$). Let $0 \leqslant r_0 < \cdots < r_l < r_{l+1}$ be such that $\{r_k, 0 \leqslant k \leqslant l+1\} = \{p_i + q_j, 0 \leqslant i \leqslant m+1, 0 \leqslant j \leqslant n+1\}$, We set

$$F^{r_k} V = \sum_{p_i + q_j \geqslant r_k} F^{p_i} V_1 \otimes F^{q_j} V_2 \ (0 \leqslant k \leqslant l+1).$$

Then

$$F^{r_k} V = \sum_{p_i + q_j = r_k} F^{p_i} V_1 \otimes F^{q_j} V_2 + F^{r_{k+1}} V.$$

This gives a filtration F^rV for V:

$$0 = F^{r_{l+1}}V \subset F^{r_l}V \subset \cdots \subset F^{r_0}V = V.$$

This definition is clearly symmetric for V_1 and V_2. So we can naturally define filtrations for higher order tensor product of vector spaces as well as for symmetric products and exterior products.

Now given a finite family of filtrations $\{F_j^{p_j}V\}$, for each j defining the corresponding slope $\mu_j(V)$, we define the *joint slope* to be

$$\mu(V) = \sum_j \mu_j(V).$$

By abuse of language, we still denote the family of filtrations $\{F_j^{p_j}\}$ by $\{F^pV\}$ and we still call it filtered vector space (tensor products are operated in corresponding terms) because all of the following conclusions are true in both cases.

Proposition 5.1.2. — The following statements are true.

1. Let (X, F^pX), (Y, F^qY), (Z, F^rZ) be filtered vector spaces over K such that the sequence
$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$
is exact. Then we have
$$\mathrm{wt}(Y) = \mathrm{wt}(X) + \mathrm{wt}(Z).$$
2. Let (V_1, F^pV_1), (V_2, F^qV_2) be filtered vector spaces over K and $V = V_1 \otimes V_2$ with the induced filtration. Then
$$\mu(V) = \mu(V_1) + \mu(V_2). \tag{5.1.2.1}$$

Proof. — (1) is clear. We now prove (2).
Since $V_1 \cong \oplus_i F^{p_i}V_1/F^{p_{i+1}}V_1$ and $V_2 \cong \oplus_j F^{q_j}V_2/F^{q_{j+1}}V_2$, $V \cong \oplus_{i,j} F^{p_i}V_1/F^{p_{i+1}}V_1 \otimes F^{q_j}V_2/F^{q_{j+1}}V_2$. Hence $F^{r_k}V/F^{r_{k+1}}V \cong \oplus_{p_i+q_j=r_k} F^{p_i}V_1/F^{p_{i+1}}V_1 \otimes F^{q_j}V_2/F^{q_{j+1}}V_2$. So we have

$$\begin{aligned}
\mu(V) &= \frac{1}{\dim V}\sum_{k=1}^{l+1} r_k \dim(F^{r_k}V/F^{r_{k+1}}) \\
&= \frac{1}{\dim V}\sum_{k=1}^{l+1}\sum_{p_i+q_j=r_k}(p_i+q_j)\dim(F^{p_i}V_1/F^{p_{i+1}}V_1)\dim(F^{q_j}V_2/F^{q_{j+1}}V_2) \\
&= \frac{1}{\dim V_1 \dim V_2}\left(\dim V_1 \,\mathrm{wt}(V_2) + \dim V_2 \,\mathrm{wt}(V_1)\right) \\
&= \mu(V_1) + \mu(V_2).
\end{aligned}$$

□

Definition 5.1.3. — The vector space V is said to be *semistable (jointly semistable)* if for any non-zero subspace $W \subset V$ with induced filtration, we have

$$\mu(V) \geqslant \mu(W).$$

An easy observation tells us that if the family consists of only one member, then semistability simply means that the filtration is trivial. Namely, the filtration is the following one:

$$0 \subsetneq V;\ 0 \leqslant p_0 < p_1.$$

Proposition 5.1.4. — For any non-zero subspace $W \subsetneq V$, we have the following equivalence relations:

$$\mu(W) \leqslant \mu(V) \Longleftrightarrow \mu(V) \leqslant \mu(V/W) \Longleftrightarrow \mu(W) \leqslant \mu(V/W).$$

And the statement remains true if we replace all "$\leqslant$" by "$<$" or "$\geqslant$".

Proof. — From the exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

we see that

$$\dim(W) + \dim(V/W) = \dim(V),\ \mathrm{wt}(W) + \mathrm{wt}(V/W) = \mathrm{wt}(V).$$

Then these relations follow from the following equivalent inequalities:

$$\frac{\mathrm{wt}(W)}{\dim(W)} \leqslant \frac{\mathrm{wt}(V)}{\dim(V)} \Longleftrightarrow \frac{\mathrm{wt}(V)}{\dim(V)} \leqslant \frac{\mathrm{wt}(V/W)}{\dim(V/W)} \Longleftrightarrow \frac{\mathrm{wt}(W)}{\dim(W)} \leqslant \frac{\mathrm{wt}(V/W)}{\dim(V/W)}.$$

□

Note that as a function, μ can have only finitely many values. For any non-zero subspace W of V, let

$$\mu_m(W) = \max\{\mu(M) : M \text{ is a non-zero subspace of } W\}.$$

And denote $\mathcal{M}(W)$ to be the set of non-zero subspaces M of W such that $\mu(M) = \mu_m(W)$. $\mathcal{M}(E)$ is partially ordered by inclusion, and it has a maximal element. Any such maximal element is semistable.

Theorem 5.1.5. — *The maximal element in $\mathcal{M}(W)$ is unique, we denote it by W_m. That is, for any $M \in \mathcal{M}(W)$, we have $M \subset W_m$.*

Proof. — When $\mu(W) = \mu_m(W)$, this is trivial, so we assume $W \notin \mathcal{M}(W)$.
We fix a maximal element $W_0 \in \mathcal{M}(W)$. We choose a non-zero subspace F of W such that $F/W_0 \in \mathcal{M}(W/W_0)$. We have the exact sequence

$$0 \longrightarrow W_0 \longrightarrow F \longrightarrow F/W_0 \longrightarrow 0.$$

Since $\dim(F) > \dim(W_0)$, we have $\mu(W_0) > \mu(F)$. So by the proposition above

$$\mu_m(W/W_0) = \mu(F/W_0) < \mu(W_0) = \mu_m(W). \qquad (\star)$$

For any $W_1 \in \mathcal{M}(W)$ distinct from W_0, let ϕ be the composition of the maps $W_1 \hookrightarrow W \twoheadrightarrow W/W_0$. We claim that $\operatorname{Ker}\phi = W_1$ hence $\operatorname{Im}\phi = 0$. First we have $\operatorname{Ker}\phi \neq 0$. Otherwise we would have $\mu(W_1) = \mu_m(W) \leqslant \mu_m(W/W_0)$. This contradicts $(\star)$. Suppose that $\ker\phi \subsetneq W_1$. From the exact sequence

$$0 \longrightarrow \ker\phi \longrightarrow W_1 \longrightarrow \operatorname{Im}\phi \longrightarrow 0$$

we have that $\mu(\operatorname{Ker}\phi) \leqslant \mu(W_1)$ hence $\mu(W_1) \leqslant \mu(\operatorname{Im}\phi)$. Since $\operatorname{Im}\phi$ is a non-zero subspace of W/W_0, we have $\mu(\operatorname{Im}\phi) \leqslant \mu_m(W/W_0)$. We get $\mu_m(W) = \mu(W_1) \leqslant \mu_m(W/W_0)$. This also contradicts ($\star$). So $\phi = 0$ hence $W_1 \subset W_0$. □

The maximal element W_m of $\mathcal{M}(W)$ is sometimes called *maximal destabilizing subspace* or *first step in Harder-Narashimhan filtration* of W.

Classically the definition of slope comes from vector bundles over a algebraic curve. Suppose E is a vector bundle over a algebraic curve C. We define

$$\mu(E) = \frac{\deg(E)}{\operatorname{rank}(E)}.$$

Starting from V and its filtration F^pV, one can construct an algebraic curve with a vector bundle $\mathfrak{C}(V)$ on it such that $\mu(\mathfrak{C}(V)) = \mu(V)$. And the semistability of V is equivalent to the semistability of $\mathfrak{C}(V)$. In particular one can use the theory developed by Narashimhan and Seshadri to prove the following result.

Theorem 5.1.6. — *Let (V_1, F^pV_1), (V_2, F^qV_2) be semistable filtered vector spaces over K. Then $V = V_1 \otimes V_2$ with the induced filtration is semistable.*

For details see ([**5**], §4). Note that the way we define the filtration for tensor product of vector spaces is consistent with tensor product of vector bundles over a algebraic curve.

5.2. Faltings-Wüstholz theorem

Let $k \subset K$ be number fields, S be a finite set of places of K, and $\mathbf{P}_k^n$ be projective n-space over k. Denote $V = H^0(\mathbf{P}_k^n, \mathcal{O}(1))$ as a k-vector space and $V_K = V \otimes_k K$. For each $v \in S$, fix $(c_{i,v})_{i=0}^{n_v}$ non-negative numbers, and $L_{0,v}, \cdots, L_{n_v,v} \in H^0\left(\mathbf{P}_K^n, \mathcal{O}_{\mathbf{P}_K^n}(1)\right)$. We equip $\mathcal{O}_{\mathbf{P}_K^n}(1)$ with an adelic structure and denote by $\|\cdot\|_v$ the corresponding adelic norm. One frequently uses the standard Fubini-Study norm for archimedean places and some orthogonal ultrametric norm for non-archimedean places (for details, see [**4**] p.552). Let $0 < p_{1,v} < \cdots < p_{m_v,v}$ be such that $\{p_{j,v}, 1 \leqslant j \leqslant m_v\} = \{c_{i,v}, i = 0, \cdots, n_v\} \setminus \{0\}$ For all $p_{j,v} > 0$, we define

$$W^{p_{j,v}} = \operatorname{Vect}\{L_{i,v}, c_{i,v} \geqslant p_{j,v}\}$$

as a vector subspace of V_K and take by convention $W^{p_{0,v}} = V_K$, $W^{p_{m_v+1,v}} = 0$. we put $p_{0,v} = 0$, $p_{m_v+1,v} = p_{m_v,v} + 1$ into $\{p_{j,v}\}_{j=1}^{m_v}$. Thus we get a new strictly increasing sequence $\{p_{j,v}\}_{j=0}^{m_v+1}$. In this way we get a filtration for V_K:

$$0 = W^{p_{m_v+1,v}} \subset W^{p_{m_v,v}} \subset \cdots W^{p_{1,v}} \subset W^{p_{0,v}} = V_K;$$

$$0 = p_{0,v} < p_{1,v} < \cdots < p_{m_v,v} < p_{m_v+1,v}.$$

For all $v \notin S$, we associate to it the trivial filtration given by

$$0 = W^{p_{1,v}} \subset W^{p_{0,v}} = V_K$$

with $\{p_{j,v}\}_{j=0}^{m_v+1} = \{0,1\}$. So we have defined a family of filtrations $\{(V_K, W^{p_{j,v}})\}_v$ for V. For each v, we can now define, as before, the slope

$$\mu_v = \mu_v(V) = \frac{\mathrm{wt}_v(V)}{\dim(V)} = \frac{1}{\dim V}\sum_{i=0}^{m} p_{j,v}\dim(W^{p_{j,v}}/W^{p_{j,v}+1})$$

and the *joint slope*

$$\bar{\mu}(V) = \sum_{v\in\mathcal{M}_K} \mu_v(V).$$

Since $\bar{\mu}(V) = 0$ for all $v \notin S$, this sum is well defined. For any subspace U of V, we equip $U_K = U \otimes_k K$ with the induced filtrations from V_K. We say that the family of filtrations is *jointly semistable* if for each nonzero subspace U of V with induced filtrations, we have

$$\bar{\mu}(U) \leqslant \bar{\mu}(V).$$

By exactly the same argument as before, we can show that there exists a unique maximal destabilizing subspace W of V. We emphasize here that W is a k-vector space.

***Theorem 5.2.1* (Faltings-Wüstholz).** — *With notation as above,*

1. *Assume that V with family of filtrations is jointly semistable, and that*
$$\bar{\mu}(V) > [K : \mathbf{Q}]. \qquad (5.2.7.1)$$
Then there exist only finitely many points $x \in \mathbf{P}^n_k(k)$ such that
$$\|L_{i,v}\|_v(x) < H_{\mathcal{O}(1)}(x)^{-c_{i,v}}. \qquad (5.2.7.2)$$
2. *Assume that $W \subset V$ is the maximal destabilizing subspace with induced filtrations, and that*
$$\bar{\mu}(W) > [K : \mathbf{Q}]. \qquad (5.2.7.3)$$
Then there exist only finitely many points $x \in \mathbf{P}^n_k(k) \smallsetminus \mathbf{P}(V/W)(k)$ satisfying (5.2.7.2).

In other words, if the case 2 happens, then the solutions to $(**)$ accumulate in subspace V/W provided that the number of them is infinite. This subspace $\mathbf{P}(V/W)$ is sometimes called the *exceptional space*. It is given by the base locus of sections in W. We shall see later that if we want to approximate a point by sequence of points in $X(k)$, then almost all elements of this sequence lie in the exceptional space. This means that the approximation constants can be computed along that subspace.

***Corollary 5.2.2* (Schlickewei).** — For any $\delta > 0$, there exist only finitely many points $x \in \mathbf{P}^n_k(k)$ which are not in $\mathbf{P}(V/W)(k)$ satisfying

$$\prod_{v\in S}\prod_{i=0}^{n_v} ||L_{i,v}||_v(x) < \frac{1}{H_{\mathcal{O}(1)}(x)^{(n+1)[k:\mathbf{Q}]+\delta}}.$$

Compare to (2.6.1), this is a generalization of the Schmidt subspace theorem (Taking into account of the adelic norm).

Remark 5.2.3. — The formula (5.2.7.1) is sharp in the sense that (5.2.7.2) always has infinitely many solutions if $\bar{\mu}(V) < [K : \mathbf{Q}]$ even though V itself is semistable. See an argument in [**13**].

We only try to give a rough overview of the proof of this powerful theorem. For the detailed proof, we refer to [**5**]. For simplicity, we assume $k = \mathbf{Q}$, $S = \{v\}$ consists of only one place v (in this case "jointly semistable is nothing but "semistable") and that $\{L_i = L_{i,v}\}$ form a basis for $V = H^0\left(\mathbf{P}^n_k, \mathcal{O}(1)\right)$. After some linear transformation, we can assume that $L_i = X_i$ are coordinate functions. We make use of the results we have developed in the previous sections. Fix integers $m, d \in \mathbf{N}$, Let $(\mathbf{P}^n)^m = \mathbf{P}^n \times \cdots \times \mathbf{P}^n$ with m factors and let π_i denote the i-th projection. Fix multi-index $D = (d_1, \cdots, d_m)$, let $\mathcal{O}(D) = \otimes_{i=1}^m \pi^* \mathcal{O}_{\mathbf{P}^n}(d_i)$. We would like to define filtrations on $H^0\left(\mathbf{P}^n_k, \mathcal{O}(d)\right)$ and $H^0\left((\mathbf{P}^n)^m, \mathcal{O}(D)\right)$. Notice that

$$H^0\left(\mathbf{P}^n_k, \mathcal{O}(d)\right) = \mathrm{Sym}^d_k V, \tag{5.2.7.4}$$

which is a quotient of $H^0\left(\mathbf{P}^n_k, \mathcal{O}(1)\right)^{\otimes d}$, and that

$$H^0\left((\mathbf{P}^n)^m, \mathcal{O}(D)\right) = \bigotimes_{j=1}^m H^0\left(\mathbf{P}^n_k, \mathcal{O}(d_j)\right) = \bigotimes_{j=1}^m \mathrm{Sym}^{d_j}_k V. \tag{5.2.7.5}$$

There are natural filtrations on them which are induced by the filtration on V introduced before the theorem. More precisely, For any monomial $M = X_0^{a_0} \cdots X_n^{a_n}$ with $\sum_{i=0}^n a_i = d$, we define the weight of M to be

$$\mathrm{wt}(M) = \sum_{i=0}^n a_i c_i.$$

For $\sigma \in H^0\left(\mathbf{P}^n_k, \mathcal{O}(d)\right)$, write $\sigma = \sum_I A_I M_I$ where M_I are monomials. Then we define the weight of σ to be

$$\mathrm{wt}(\sigma) = \min\left\{\mathrm{wt}(M_I) : A_I \neq 0\right\}.$$

Denote $V_d = H^0\left(\mathbf{P}^n_k, \mathcal{O}(d)\right)$ then we define the filtration characterized by weight as follows:

$$F^q(V_d) = \{\sigma \in V_d : \mathrm{wt}(\sigma) \geqslant q\} \quad \text{for } q > 0.$$

Then, as before, if we choose $\{p_{i,d}\} = \{\sum_{i=0}^n a_i c_i\}$ with $\sum_{i=0}^n a_i = d$ and make canonical constructions, we easily check that this filtration on $H^0\left(\mathbf{P}^n_k, \mathcal{O}(d)\right)$ agrees with the induced one. Next denote $V_D = H^0\left((\mathbf{P}^n)^m, \mathcal{O}(D)\right)$. For $\phi \in V_D$, write $\phi = \sum_J B_J \bigotimes_{j=1}^m \pi_j^* M_j^J$ with $M_j^J \in V_{d_j}$. We define the weight of ϕ to be

$$\mathrm{wt}(\phi) = \min\left\{\sum_{j=1}^m \mathrm{wt}(M_j^I) : B_J \neq 0\right\}.$$

We then also get a filtration on V_D, which coincides with the induced one. So we can compute the slopes $\mu(V_d)$ and $\mu(V_D)$. To apply these ideas, assume the contrary of case 1. Then we can find rational points $(x_i)_{i=1}^m$ with arbitrarily big height which violate $(**)$.

$$0 \ll H(x_1) \ll \cdots \ll H(x_m).$$

we choose integers $\{d_i\}$ such that for some $A \gg 0$,

$$d_i \simeq \frac{A}{\log H(x_i)}.$$

Hence

$$d_j/d_{j+1} \gg 1.$$

Let $x = (x_1, \cdots, x_m)$ be the rational point in $(\mathbf{P}^n)^m(k)$. If $\phi \in \mathcal{O}(D)$ does not vanish at $x \in \mathbf{P}$, we can easily get un lower bound for ϕ at $\mathcal{O}(D)(x)$ in term of the adelic norm. If ϕ vanishes at x, we can define *differential operators* (see [**5**], 7) to get the desired lower bound estimation. This is similar to the step 1 of the proof of the Roth's theorem. The other rough part is to construct such ϕ which vanishes at x at high order. In [**5**], they use the **Product theorem** of Faltings (see [**4**] p. 561) to locate x in a product subvariety Z, $Z = Z_1 \times \cdots Z_m$ with Z_i a closed subvariety of $\mathbf{P}^n_k$, which is an irreducible component of a closed subscheme consisting of points of high vanishing order.

Theorem 5.2.4. — *Fix ϕ. Choose a positive number ε. Suppose there exists an r such that*

$$d_1/d_2 \geqslant r, \cdots, d_{m-1}/d_m \geqslant r$$

and that Z is an irreducible component of $Z_{\sigma+\varepsilon}$ which is also an irreducible component of Z_σ. (Here Z_σ denotes closed the subscheme of $(\mathbf{P}^n)^m$ where the index *is $\geqslant \sigma$.) Then we have the following conclusions:*

1. *$Z = Z_1 \times \cdots Z_m$ is a product variety;*
2. *The degrees of Z_i are bounded by some constant depending on ε.*

(For the definition of *index* and a proof of this theorem, we refer to ([**4**] 3.1).) The problem reduces to finding some nonzero polynomial $\varphi \in H^0(Z, \mathcal{O}_Z(D)$ satisfying some vanishing condition. For any subvariety Y of $\mathbf{P}^n_k$, the classical vanishing theorem shows that for all $d \gg 0$, the natural restriction map

$$H^0\left(\mathbf{P}^n_k, \mathcal{O}(d)\right) \longrightarrow H^0(Y, \mathcal{O}_Y(d))$$

is surjective. Hence $H^0(Y, \mathcal{O}_Y(d))$ is a quotient of $H^0\left(\mathbf{P}^n_k, \mathcal{O}(d)\right)$ and, in the view of the formula (5.2.7.4), is also a quotient of $H^0\left(\mathbf{P}^n_k, \mathcal{O}(1)\right)^{\otimes d}$. Consequently we can form an induced quotient filtration on $H^0(Y, \mathcal{O}_Y(d))$ and compute the slope $\mu_d(Y)$. So in this way we can form an induced filtration on the product variety Z, which is naturally a quotient of the filtration on V_D.

We define the *expectation value*

$$E(Y) = \lim_{d \to \infty} \frac{\mu_d(Y)}{d}.$$

We can also define it similarly for subvarieties in $(\mathbf{P}^n)^m$. Notice that if $X = X_1 \times \cdots \times X_m$ is a product subvariety, then

$$E(X) = \sum_{j=1}^{m} E(X_j).$$

We can show that (see [**5**] §5 using probabilistic methods) this limit exists. And it measures the asymptotic average slope hence is like an "expectation value". Then the problem reduces to proving that $E(Z)$ is bigger than some certain value C and in that case one can surely find one desired auxillary polynomial. In fact by computation, formula (5.2.7.1) means that $m\mu(V) \geqslant C$. By theorem (5.1.6), the filtration on $V \otimes d$ is semistable for any d. By proposition (5.1.4), we have, using the notations above, for all $d \gg 0$,

$$\frac{\mu(H^0(Y, \mathcal{O}_Y(d)))}{d} \geqslant \frac{\mu(V^{\otimes d})}{d} = \mu(V).$$

Then

$$\begin{aligned} E(Y) &= \lim_{d \to \infty} \frac{\mu(H^0(Y, \mathcal{O}_Y(d)))}{d} \\ &\geqslant \lim_{d \to \infty} \frac{\mu(V^{\otimes d})}{d} \\ &= \mu(V) \end{aligned}$$

Apply this to the product variety Z we get

$$E(Z) = \sum_{j=1}^{m} E(Z_j) \geqslant m\mu(V) \geqslant C.$$

which is exactly what we want.

In case 2 where $W \subset V$ is the first step of the Harder-Narashimhan filtration, W equipped with the induced filtration is (jointly) semistable. We write $V = W \oplus U$ with U a complementary subspace to W. Let

$$\pi : \mathbf{P}(V) \dashrightarrow \mathbf{P}(W) = \mathbf{P}(V/U)$$

be the linear projection. Denote $n+1 = \dim V$, $s+1 = \dim W$. The projection map can be viewed in terms of coordinates

$$\pi : [x_0, \cdots, x_n] \to [x_0, \cdots, x_s].$$

So the domain of definition of π is the complement of

$$\{[x_0, \cdots, x_n] \in \mathbf{P}(V) : x_0 = \cdots = x_s = 0\},$$

namely the linear subvariety

$$\Lambda = \mathbf{P}(V/W),$$

which is also viewed as the base locus of $W \subset \mathcal{O}_{\mathbf{P}(V)}(1)$. Now we want to reduce the problem to the space $\mathbf{P}(W)$. Taking into account of (5.2.7.3), we can find some linear forms $\{G_j\}$ in the induced filtration of W such that $\{G_j\}$ together with $\{e_j\}$ (a subset of $\{c_i\}$ where at least one $e_j > 0$) generate the same filtration on W as the induced one and that $\{G_j\}$ are in the following forms

$$G_j = \sum_i a_{ij} L_i \ (a_{ij} = 0 \text{ unless } e_j \geqslant c_i).$$

We denote by A the set of solutions to (5.2.7.2) which do not lie in $\mathbf{P}(V/W)(k)$. Then there exists a constant c (depending on a_{ij} and the number of $\{L_i\}$) such that for any $x \in A$,

$$\|G_j\|(x) \leqslant cH_{\mathcal{O}_{\mathbf{P}(V)}(1)}(x)^{-e_j};\ \forall j. \tag{5.2.7.6}$$

By the case that we have proven before, the set of solutions to

$$\|G_j\|(y) \leqslant cH_{\mathcal{O}_{\mathbf{P}(W)}(1)}(y)^{-e_j};\ \forall j \tag{5.2.7.7}$$

is finite. Since $H_{\mathcal{O}_{\mathbf{P}(V)}(1)}(x) \geqslant H_{\mathcal{O}_{\mathbf{P}(W)}(1)}(\pi(x))$ and $G_j(x) = G_j(\pi(x))$, we conclude that the set A is mapped under projection to a finite set B which is contained the set of solutions to (5.2.7.7). For any $y \in B$, the fiber $\pi^{-1}(y) \cap A$ must be finite. Otherwise there exist a sequence of points $\{x_l\}$ in $\pi^{-1}(y) \cap A$ such that $H_{\mathcal{O}_{\mathbf{P}(V)}(1)}(x_l) \to \infty$. Then for all $c_i > 0$, $L_i(x_l) \to 0$. It follows that for any j, $G_j(y) \equiv G_j(x_l) \to 0$ hence $G_j(y) = 0$. But one can always find $e_j > 0$ such that G_j does not vanish at y. Indeed, suppose for all $e_j > 0$, we have $G_j(y) = 0$. Denote Y the subspace generated by those G_j, we have $Y \subsetneq W$ (otherwise y would be a base point of W, which is impossible because $W = \mathcal{O}(1)$ is invertible). Then with the induced filtration, $0 < \mu(W) \leqslant \mu(W/Y) = 0$ since W is semistable. This gives a contradiction. So we conclude that $\pi^{-1}(y) \cap A$ is finite for every $y \in B$. So A is finite.

CHAPTER 6

MAIN THEOREM

6.1. Main theorem

Let k be a number field, L be a $\mathbf{Q}$-ample line bundle on X, and S be a finite set of places of k. For each $v \in S$, we fix an extension of v to $\bar{k}$ and an $x_v \in X(\bar{k})$. We now state a simultaneous approximation theorem which generalizes many classical approximation theorems listed before.

***Theorem 6.1.1* ([10], 5.1)**. — *Let $\{R_v\}_{v\in S}$ be positive real numbers such that*

$$\sum_{v\in S} \beta(L, x_v) R_v > 1. \tag{6.1.1.1}$$

Then there exists a proper subvariety Z of X such that the following two equivalent statements are true:

1. *For all sequences of pairwise distinct points (x_i) in $X(k) \smallsetminus Z(k)$, there exists $v \in S$ such that $\alpha_{x_v}^{v,L}((x_i)) \geqslant R_v^{-1}$.*
2. *For any set of strictly positive real numbers $\{\delta_v\}_{v\in S}$, there exist only finitely many points $x \in X(k) \smallsetminus Z(k)$ satisfying*

$$d_v(x_v, x) \leqslant \frac{1}{H_L(x)^{R_v+\delta_v}};\ v \in S.$$

We emphasize here that the proper subvariety Z can be empty. This would happen for the trivial example where we take each R_v larger than $\alpha_x^v(L)$. We shall discuss more about this after the proof.

Without specifying the exceptional subvariety, we can also state a slightly different conclusion allowing equality in (6.1.1.1).

Corollary 6.1.2. — Let $\{R_v\}_{v\in S}$ be positive real numbers such that

$$\sum_{v\in S} \beta(L, x) R_v \geqslant 1.$$

Then for any strictly positive real numbers $\{\delta_v\}_{v\in S}$, there exists a proper subvariety Z of X such that there exist only finitely many points in $X(k)\smallsetminus Z(k)$ satisfying

$$d_v(x_v,x)\leqslant \frac{1}{H_L(x)^{R_v+\delta_v}};\ v\in S.$$

To show that the theorem implies the corollary, just set $R'_v=R_v+\delta_v/2, \delta'_v=\delta_v/2$ and apply the theorem.

Sketch of the proof of the theorem. — The pivotal tool is the Faltings-Wüstholz theorem. We first make some preparations.
We fix K a field of definition for all x_v. For each $v\in S$, denote by $\pi_v:\widetilde{X}^v\to X_K$ the blow up along the point x_v, by E^v the exceptional divisor and by $L^v_\gamma=\pi_v^*L_K-\gamma E^v$. The density function $f_v(\gamma)$ and the approximation constant $\beta(L,x_v)$ are computed over K. (Note that the base change K/k is flat.) We may choose rational numbers $(\gamma_i^v)_{i=1}^{r_v}$ such that

$$0<\gamma_i^v<\cdots<\gamma_{r_v}^v<\gamma_e^v,$$

and that

$$\sum_{v\in S}R_v\left(\sum_{i=1}^{r_v}\gamma_i^v(f_v(\gamma_i^v)-f_v(\gamma_{i+1}^v))\right)>1 \tag{G}$$

because of (6.1.1.1).
We now fix an sufficiently large integer m such that:
•(H1) mL_K is integral and very ample;
•(H2) $m\gamma_i^v$ are integers for all i and v;
•(H3) (y the definition of volume) $h^0(mL_{\gamma_i^v})/h^0(mL_K)=h^0(mL_{\gamma_i^v})/h^0(mL)$ is sufficiently close to $f_\gamma(\gamma_i^v)$ for all i and v such that

$$\sum_{v\in S}\frac{R_v}{h^0(mL)}\left(\gamma_i^v(h^0(mL_{\gamma_i^v})-h^0(mL_{\gamma_{i+1}^v}))\right)>1. \tag{$*$}$$

We are ready to introduce filtrations. Fix $v\in S$. Let $V=\Gamma(X,mL)$. Then $V_K=V\otimes_k K=\Gamma(\widetilde{X}^v,mL_0^v)$. We embed X_K into the projective space $\mathbf{P}(V_K)$ over K via mL so V_K is the space of sections obtained by pulling back of global sections of $\mathcal{O}(1)$ on $\mathbf{P}(V_K)$. Set $F^iV_K=\Gamma(\widetilde{X}^v,mL_{\gamma_i^v})$ and $c_i^v=R_v\gamma_i^v[K:\mathbf{Q}]$. Choose a basis $\{s_i^v\}_{i=1}^{n_v}$ for V_K which is compatible with F^iV_K. This means that for all $j\leqslant n_v$,

$$\mathrm{Vect}_K\{s_i^v,i\geqslant j\}=F^jV_K.$$

So we get a filtration for V_{K^v} (compare this with the construction in Chap. 5 §2):

$$0=F^{n_v+1}V_K\subset F^{n_v}V_K\subset\cdots\subset F^1V_K\subset F^0V_K=V_K;$$

$$0=c_0^v<c_1^v<\cdots<c_{n_v}^v<c_{n_v+1}^v.$$

Inequality ($*$) implies

$$\sum_{v\in S}\frac{1}{\dim V_K}\left(c_i^v(\dim_K F^iV_K-\dim_K F^{i+1}V_K)\right)=\sum_{v\in S}\mu_v(V)>[K:\mathbf{Q}],$$

which is exactly the hypothesis (5.2.7.1). By the Faltings-Wüstholz theorem, we conclude that there exist a proper subvariety Z of X and that there exist only finitely many points x in $X(k)\smallsetminus Z(k)$ satisfying

$$||s_i^v||(x) < H_{mL}(x)^{-c_i^v};\ v\in S,\ 1\leqslant i\leqslant n_v. \qquad (**)$$

We can now prove the theorem. We assume the contrary holds. Namely, there exists a sequence (x_i) of pairwise distinct points in $X(k)\smallsetminus Z(k)$ such that $\alpha_{x_v}^{v,L}((x_i)) < R_v^{-1}$ for all $v\in S$. This means that $d_v(x_i,x_v)\to 0$ for all v and that for all sufficiently small $\varepsilon>0$ such that $\xi_v = R_v^{-1}-\varepsilon > \alpha_{x_v}^{v,L}((x_i))$ for all $v\in S$, $d_v(x_i,x_v)^{\xi_v}H_L(x_i)\to 0$. On the other hand, by (1.1.4), for each v, every s_i^v vanishes at x_v at order $\geqslant m\gamma_i^v$. Since by (3.3.2) locally distance functions are equivalent to the maximum of the absolute value of local equations (valuated at the sequence). Set $\delta_v=\xi_v(m\gamma_i^v[K:\mathbf{Q}])^{-1}>0$.We see that for all i large enough, $||s_i^v||(x_i)\leqslant Cd_v(x_i,x_v)^{m\gamma_i^v[K:\mathbf{Q}]-\delta_v}$. Then we have

$$||s_i^v||(x_i)^{(mc_i^v)^{-1}}H_L(x_i)\leqslant d_v(x_i,x_v)^{R_v^{-1}-\delta_v}H_L(x_i)\to 0;\quad \text{for all } v\in S.$$

This contradicts $(**)$. □

Remark 6.1.3. — To fill in a point we have left before the proof, we see that the larger each R_v is, the more flexibility of choosing the sequence (γ_v) we have. And the more possibility we have to make the vector space $V=\Gamma(X,mL)$ itself to be semistable. In this case the zero locus Z is empty. So what would be interesting is that when each R_v is sufficient small such that $R_v^{-1} > \alpha_x^v(L)$ (of course they must also satisfy the hypothesis (6.1.1.1)). Then, as the theorem implies, the vector space V is not semistable anymore. Hence the zero locus Z of the maximal destabilizing subspace is non-empty. And it contains almost all k-points that are "quite near" x (that is, sequences with smaller approximation constant α).

We will see some applications in the next chapter.

***Remark 6.1.4* (Possible generalizations)**. — A priori the exceptional subvariety is not necessarily unique. It is determined, as in the proof above, by the sequence and integer chosen in order to satisfy hypothesis (G) and (H1-3). We can make some refinement of it, using a method similar to the construction of "asymptotic base locus".

Construction. — We fix a sequence of sets $\{D_r\}$ where each D_r consists of finitely many rational numbers $\{\gamma_{i,v}^r\}$ satisfying the hypothesis (G) and that the formula on the right hand side tends to $\sum_{v\in S}\beta(x,L)R_v$. For each r, let a_r be the smallest integer such that hypothesizes (H1-3) are satisfied. Then as in the proof, for each integer na_r, we get an exceptional subvariety $Z_{n,r}$ (defined as the base locus of the maximal destabilizing subspace of $\Gamma(X,na_rL)$). Now let

$$Z_r^l=\left(\bigcap_{n=1}^{l} Z_{n,r}\right)_{\text{red}}\quad \text{(intersection as sets)}.$$

We get a decreasing sequence of closed subvarieties $\{Z_r^l\}$. By Noetherianity there exists an integer b_r such that for any $l \geqslant b_r$, $Z_r^l = Z_r^{b_r}$. Since

$$X(k) \smallsetminus Z_r^l(k) \subseteq \bigcup_{n=1}^{l} X(k) \smallsetminus Z_{n,r}(k),$$

It follows that Theorem 6.1.1 holds with respect to the exceptional subvariety $Z_r^{b_r}$. Furthermore, let

$$Z_m = \left(\bigcap_{r=1}^{m} Z_r^{b_r} \right)_{\mathrm{red}} \quad \text{(intersection as sets)}.$$

Also by Noetherianity there exists an integer s such that for any $m \geqslant s$, $Z_m = Z_s$. And Theorem 6.1.1 holds with respect to Z_s. Of course there are still other ways to make the exceptional subvariety smaller. □

CHAPTER 7

SOME APPLICATIONS

7.1. Generalizations of classical approximation theorems

We continue using the notation as in Chapter 6.

Definition 7.1.1. — Let $x \in X(\bar{k})$, Z be a non-empty proper subvariety of X, L be an ample line bundle on X and $v \in \mathcal{M}_k$. We say that the approximation constant $\alpha_x^v(L)$ *is computed over* Z if there exists $\varepsilon_0 > 0$ such that for all sequence (x_i) of k-points in $X(k)$ with $\alpha_x^{v,L}((x_i)) - \alpha_x^v(L) < \varepsilon_0$, all but finitely many elements of the sequence are in $Z(k)$. In this case we sometimes call Z a *exceptional subvariety.*

Definition 7.1.2. — With the same notation as before, we say that the approximation constant $\alpha_x^v(L)$ *can computed over* Z if $\alpha_x^v(L) = \alpha_x^v(L|_Z)$.

Intuitively "$\alpha_x^v(L)$ is computed over Z" means that almost all k-points which are sufficiently near x accumulate on the subvariety Z. In particular $\alpha_x^v(L) = \alpha_x^v(L|_Z)$, namely "is computed" is stronger than "can be computed".

The first theorem as an application is a rephrased version of Theorem 6.1.1 in terms of the definition above.

Theorem 7.1.3. — *Let L be an ample $\mathbf{Q}$-line bundle, v be a place of k and $x \in X(\bar{k})$. Then one and only one of two statements below is true:*

1. $\alpha_x^v(L) \geqslant \beta(L, x)$*;*
2. *There exists a proper subvariety Z of X such that $x \in Z(\bar{k})$ and $\alpha_x^v(L)$ is computed over Z.*

Proof. — Suppose $\alpha_x^v(L) < \beta(L, x)$. Choose $\varepsilon_0 \in]\alpha_x^v(L), \beta(L, x)[$. By Theorem 6.1.1 applied to $R_v = \varepsilon_0^{-1}$ we get that there exists a non-empty proper subvariety Z of X such that for all sequence of k-points (x_i) with $\alpha_x^{v,L}((x_i)) < \varepsilon_0$, all but finitely of them lie in $Z(k)$. □

Remark 7.1.4. — The exceptional subvariety Z just obtained from the proof depends on the choice of ε_0. Like what we have done in Remark 6.1.4, we may even make the exceptional subvariety Z smaller by taking a sequence $(\varepsilon_{0,i})$ tending to $\alpha_x^v(L)$, making intersection of the exceptional subvarieties obtained each time in Remark 6.1.4 and using Noetherianity.

We can now state some generalizations of the classical approximation theorems in Chapter 2.

***Theorem 7.1.5* (Schmidt).** — *Let L be an ample $\mathbf{Q}$-line bundle, v be a place of k and $x \in X(\bar{k})$. Then one and only one of two statements below is true:*

1. $\alpha_x^v(L) \geqslant \frac{n}{n+1}\varepsilon(L,x)$*;*
2. *There exists a proper subvariety Z of X such that $x \in Z(\bar{k})$ and $\alpha_x^v(L)$ is computed over Z.*

This follows directly from Theorem 7.1.3 and the inequality in Theorem 4.2.4.

We see that this theorem generalizes the Siegel-Thue-Schmidt theorem (Corollary 2.6.2) in Chap. 2 (as the special case of $\varepsilon(L,x) = \varepsilon(\mathcal{O}(1),x) = 1$).

***Theorem 7.1.6* (Roth).** — *With the notation above, we have $\alpha_x^v(L) \geqslant \frac{1}{2}\varepsilon(L,x)$.*

In fact, case (1) in Theorem 7.1.5 is stronger than the statement here. For case (2), the problem is reduced to a proper subvariety Z with lower dimension. So we can apply the theorem again for the variety Z and notice that the Seshadri constant computed through a subvariety is no smaller than that computed through X. Then we get the desired conclusion. We refer to ([**10**] 2.16, 6.3) for a discussion of the criterion of obtaining the equality.

We mention a generalized Liouville's theorem appearing in ([**11**]) obtained using a different approach.

***Corollary 7.1.7* (Liouville).** — With the notation above, moreover set $d = [K : k]$ where K is the field of definition of x. Then we have $\alpha_x^v(L) \geqslant \frac{1}{d}\varepsilon(L,x)$.

***Example 7.1.8* ($\mathbf{P}^1 \times \mathbf{P}^1$ continued).** — We continue using the notation of Section 4.3. We have $\alpha_x^v(\mathcal{O}(a,b)) = \min(a,b) < \frac{a+b}{2} = \beta(\mathcal{O}(a,b),x)$. So by Theorem 7.1.3 we are in situation (2). To find a suitable exceptional subvariety Z, doing computation as in the proof of Theorem 6.1.1 depends on the choice of ε_0 and sequence and integer satisfying (H1-3) and (G). Also finding the maximal destabilizing subspace is not so easy. But one observes that if we suppose $x = [1:0] \times [1:0]$ and $a \leqslant b$, then if we approximate x through the rational curve $\mathbf{P}^1 \times [1:0]$ we are doing the same thing as approximating the point $[1:0]$ in $\mathbf{P}^1$ in terms of the line bundle $\mathcal{O}(a)$ and we get the same approximation constant. Thus $\alpha_x^v(\mathcal{O}(a,b))$ **can be computed** through this rational curve.

7.2. Application to simultaneous approximation

Next we are going to extend Theorem 6.1.1 to simultaneous approximation of finitely many points. This also generalizes Theorem 2.5.1. We shall use this to deduce a very special case of Vojta's inequality.

Let S be a finite set. We call a function $\xi : S \to [0,1]$ to be a *weight function* if $\sum_{v\in S} \xi(v) = 1$. We begin with a technical lemma, which helps us reduce the verification for possibly infinitely many weight functions to a finite number.

Lemma 7.2.1. — Let $\{A_v\}_{v\in S}$ and $\{B_v\}_{v\in S}$ be sets of positive real numbers with $A_v < B_v$ for all $v \in S$. Then there exists a finite set Ω of weight functions such that for any function $\zeta : S \to \mathbf{R}_{\geqslant 0}$ satisfying $\sum_{v\in S} \zeta(v) \geqslant 1$, there exists a weight function $\xi \in \Omega$ satisfying $\xi(v)A_v \leqslant \zeta(v)B_v$ for all $v \in S$.

Remark 7.2.2. — Lemma 7.2.1 would fail to be valid if we replace $A_v < B_v$ by the weaker assumption $A_v \leqslant B_v$.

Proof. — We fix an integer N such that

$$\min_{v\in S} \frac{B_v}{A_v} - \frac{\mathrm{Card}(S)}{N} \geqslant 1.$$

(This integer N exists because of the assumption of $A_v < B_v$.) We set Ω to be the set of all weight functions ξ satisfying that $N\xi$ is an integer valued function. We now verify that this set satisfies the property we want. Fix $\zeta : T \to \mathbf{R}_{\geqslant 0}$ with $\sum_{v\in S} \zeta(v) \geqslant 1$, define a function ϕ as follows:

$$\phi(v) = \lfloor \frac{N\zeta(v)B_v}{A_v} \rfloor.$$

Then ϕ is an integer valued function and $\phi(v) \geqslant \frac{N\zeta(v)B_v}{A_v} - 1$, hence

$$\sum_{v\in S} \phi(v) \geqslant \sum_{v\in S} \frac{N\zeta(v)B_v}{A_v} - 1 \geqslant \left(N \min_{v\in S} \frac{B_v}{A_v}\right) \sum_{v\in S} \zeta(v) - \mathrm{Card}\, S \geqslant N.$$

We can find some integer valued function $\psi : S \to \mathbf{N}$ such that $\psi \leqslant \phi$ and that $\sum_{v\in S} \psi(v) = N$. Then the function ξ defined as $\xi(v) = \frac{1}{N}\psi(v)$ is an element of Ω and satisfies

$$\xi(v)A_v = \frac{\psi(v)A_v}{N} \leqslant \frac{\phi(v)A_v}{N} \leqslant \zeta(v)B_v.$$

□

Theorem 7.2.3. — *We use the notation as in the beginning of Chapter 6. Let* $\{R_v\}_{v\in S}$ *be a set of positive numbers such that* $R_v > \frac{n+1}{n}\varepsilon(L, x_v)^{-1}$ *for all* $v \in S$. *Then there exists a proper subvariety* Z *of* X *such that for any* $\delta > 0$, *there exist only finitely many solutions* $x \in X(k) \smallsetminus Z(k)$ *to*

$$\prod_{v\in S} d_v(x_v, x)^{R_v^{-1}} \leqslant \frac{1}{H_L(x)^{1+\delta}}. \qquad (*)$$

To make use of Theorem 6.1.1, we shall begin to prove this theorem in a slightly different form:

Lemma 7.2.4. — It suffices to prove the following two equivalent statements:

1. There exists a proper subvariety Z of X such that for any weight function ξ, and any sequence (x_i) of pairwise distinct points in $X(k) \smallsetminus Z(k)$, there exists $v \in S$ with $\xi(v) \neq 0$ such that $\alpha_{x_v}^{v,L}((x_i)) \geqslant (R_v\xi(v))^{-1}$.

2. There exists a proper subvariety Z of X such that for any weight function ξ, and any set $\{\delta_v\}_{v\in S}$ of positive real numbers, there are only finitely many solutions $x \in X(k) \smallsetminus Z(k)$ to
$$d_v(x_v, x) \leqslant \frac{1}{H_L(x)^{R_v\xi(v)+\delta_v}}.$$

Proof of the lemma. — Assume the theorem is false. Then there exists $\delta > 0$ such that we can find a infinite sequence (x_i) of solutions to $(*)$. Since L is ample, by Northcott's theorem we may assume $H_L(x_i) > 1$ for all i (by passing to a subsequence). For each i, we define
$$\zeta_i(v) = -\frac{\log d_v(x_v, x_i)}{(1+\delta)R_v h_L(x_i)}.$$
We have $\sum_{v\in S}\zeta_i(v) \geqslant 1$ by $(*)$. Fix a positive number $\varepsilon < \delta$. Apply Lemma 7.2.1 we get a finite set Ω of weight functions such that for any i there exists $\xi \in \Omega$ satisfying
$$\xi(v)(1+\varepsilon) \leqslant \zeta_i(v)(1+\delta) \text{ for all } v \in S. \qquad (**)$$
By passing to a subsequence again, we may assume that $(**)$ holds for all $v \in S$ and all i. Then the infinite sequence (x_i) satisfies
$$d_v(x_v, x_i) = \frac{1}{H_L(x_i)^{\zeta_i(v)(1+\delta)}} < \frac{1}{H_L(x_i)^{\xi(v)(1+\varepsilon)}}.$$
This gives a contradiction to (2). □

Proof of the theorem. — We choose a set of positive real numbers $\{T_v\}_{v\in S}$ such that for all $v \in S$,
$$R_v > T_v > \frac{n}{n+1}\varepsilon(L, x_v)\left(\geqslant \beta(L, x_v)^{-1}\right).$$
Apply Lemma 7.2.1 we get a finite set Ω of weight functions such that for any function $\zeta : S \to \mathbf{R}_{\geqslant 0}$, there exists $\xi \in \Omega$ satisfying $\xi(v)T_v \leqslant \zeta(v)R_v$ for all $v \in S$. Now for each $\xi \in \Omega$, set $S(\xi) = \{v \in S : \xi(v) \neq 0\}$. Since
$$\sum_{v\in S}\xi(v)T_v\beta(L, x_v) = \sum_{v\in S(\xi)}\xi(v)T_v\beta(L, x_v) > \sum_{v\in S}\xi(v) = 1.$$
By Theorem 6.1.1, there exists a proper subvariety Z_ξ such that for any sequence of pairwise distinct points (x_i) in $X(k) \smallsetminus Z_\xi(k)$, there exists $v \in S(\xi)$ satisfying $\alpha_{x_v}^{v,L}((x_i)) \geqslant (\xi(v)T_v)^{-1}$. We define the exceptional subvariety Z as
$$Z = \bigcup_{\xi\in\Omega} Z_\xi.$$
For any weight function ζ and any sequence of distinct points $\{y_i\}$ in $X(k) \smallsetminus Z(k)$, we choose $\xi \in \Omega$ such that $\xi T_v \leqslant \zeta R_v$ for all $v \in S$. Then $\exists v \in S(\xi)$ satisfying
$$\alpha_{x_v}^{v,L}((x_i)) \geqslant (\xi(v)T_v)^{-1} \geqslant (\zeta(v)R_v)^{-1}.$$
This is exactly what need to prove for the lemma above. □

Imitating Corollary 6.1.2, we also have

Corollary 7.2.5. — Let $\{R_v\}_{v\in S}$ be a set of positive numbers such that $R_v \geqslant \frac{n+1}{n}\varepsilon(L,x_v)^{-1}$ for all $v\in S$. Then for any $\delta>0$, there exists a proper subvariety Z of X such that there exist only finitely many solutions $x\in X(k)\smallsetminus Z(k)$ to

$$\prod_{v\in S} d_v(x_v,x)^{R_v^{-1}} \leqslant \frac{1}{H_L(x)^{1+\delta}}.$$

7.3. A special case of Vojta's type inequality

We fix X to be a projective smooth variety of dimension n defined over a number field k. Denote by K_X the canonical bundle. We say that a divisor D has *normal crossing* if locally it can be expressed in local coordinates $x_1,\cdots,x_n$ of the form $x_1\cdots x_i=0$. Let $S\subset\mathcal{M}_k$ be a finite set of places. For any divisor L on X, we set $h_L(\cdot)=\log H_L(\cdot)$. Vojta's conjecture states that the growth of the height of rational points as they approach a simple normal crossing divisor. It has many striking corollaries, including the famous "abc Conjecture". We mention its general form here.

Conjecture 7.3.1. — *Let A be a big line line bundle and D be a normal crossing divisor on X. Then for any $\varepsilon>0$, there exists a closed subvariety Z_ε of X such that for all $P\in X(k)\smallsetminus Z(k)$,*

$$\sum_{v\in S}\lambda_{D,v}(P)+h_{K_X}(P)\leqslant\varepsilon h_A(P)+O(1)$$

where $\lambda_{D,v}(\cdot)$ is a local height function for D at v.

One can also formulate this kind of conjecture with respect to other subvarieties of X (other than one normal crossing divisor). In fact in the result we are going to state the subvariety "D" is just finitely many points. In this case we need to change the "local height" into distance function. This is a direct conclusion of simultaneous approximation to finitely many algebraic points we just obtained above.

Theorem 7.3.2. — *Assume that X is a Fano variety (namely $-K_X$ is ample) and that D is a finite set of points $X(\bar{k})$. Assume further more that $\varepsilon(-K_X,x)>\frac{n+1}{n}$ for all $x\in D$. Then for any $\varepsilon>0$, there exists a closed subvariety Z of X such that for all $P\in X(k)\smallsetminus Z(k)$, we have*

$$\sum_{v\in S}-\log d_v(x,P)<(1+\delta)h_{-K_X}(P)+O(1).$$

Proof. — We apply Theorem 7.2.3 with $R_v=1>\frac{n+1}{n}\varepsilon(-K_X,x)^{-1}$ for all $v\in S$. We get that for any $\varepsilon>0$, there exists a closed subvariety Z of X such that for all but finitely many $P\in X(k)\smallsetminus Z(k)$

$$\prod_{v\in S,x\in D} d_v(x,P)>H_{-K_X}(P)^{-(1+\varepsilon)}.$$

(Of course we can make Z contain D.) We then take log on both sides and add a constant to get the desired inequality for all $P\in X(k)\smallsetminus Z(k)$. □

Corollary 7.3.3. — With the same assumptions as before. Then for any $\varepsilon > 0$, and any big line bundle A on X, there exists a closed subvariety Z of X such that for all $P \in X(k) \smallsetminus Z(k)$, we have

$$\sum_{v \in S, x \in D} -\log d_v(x, P) + h_{K_X}(P) \leqslant \varepsilon h_A(P) + O(1).$$

Proof. — By Theorem 1.2.10 we see that for any big line bundle A on X, choose integer m and effective line bundle E such that mA is linearly equivalent to $-K_X + E$. By the previous theorem applied to $\delta = \varepsilon/m$, we get that there exists a closed subvariety Z of X such that for all $P \in X(k) \smallsetminus Z(k)$,

$$\sum_{v \in S, x \in D} -\log d_v(x, P) + (1 + \delta) h_{K_X}(P) < O(1).$$

We set Z' to be the union of Z and the asymptotic base locus $B(E)$ of E. By Proposition 1.4.4, $h_E(P) > O(1)$ for $P \in X(k) \smallsetminus B(E)(k)$. We conclude that for all $P \in X(k) \smallsetminus Z'(k)$,

$$\sum_{v \in S, x \in D} -\log d_v(x, P) + (1 + \delta) h_{K_X}(P) < h_E(P) + O(1).$$

We get the desired inequality by moving and combining corresponding terms. □

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