Université Joseph Fourier

Master 2 Research Report in Automatic Control

Event-based control of linear hyperbolic systems

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Contents

Abstract

In this document, we introduce an event-based boundary control approach for 1-dimensional linear hyperbolic systems. A preliminary study of event-based control theory of finite dimensional systems was carried out. Simultaneously, classical techniques for stabilization of linear hyperbolic systems were studied. By following the main ideas for event-based already developed for finite-dimensional systems, an extension to infinite case, via Lyapunov techniques, was done. The main contribution lies in the definition of two event-triggering conditions, which turned out to be appropriate, reducing number of execution times while preserving a good level of performance. Some numerical examples are illustrated to validate the theoretical results.

Resumé

Dans ce document on introduit une approche de commande sur la frontière basée sur des événements pour des systèmes linéaires hyperboliques dans un dimension. Une étude préliminaire de cette commande a été faite pour les systèmes de dimension fini. Des études sur les techniques classiques pour le stabilisation des systèmes hyperboliques linéaires ont été faites en même temps. Puis, avec les idées principales, une extension pour le cas infini en utilisant les techniques de Lyapunov a été effectuée. La contribution principale dépend sur la définition de deux conditions de déclenchement, qui sont devenues propre, en diminuant le nombre des exécution et préservant un bon niveau de performance. Quelques exemples numériques sont illustrés avec leur résultats correspondant.

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"En mi soledad he visto cosas muy claras que no son verdad"
Antonio Machado

1 Introduction

Event-based control is a computer control strategy recently developed for finite dimensional systems aiming to efficiently use communications and computational resources. In particular, it arises in the context of Networked Control Systems. In fact, the increasing popularity of wired and wireless network control systems has led to highlight the importance of addressing energy, computational and communication constraints when defining feedback control loops [28] [15]. In addition, traditionally digital control techniques often assume that controllers execute periodically but it may result in unnecessary workloads when computational and communication resources may be devoted to other tasks [12]. Therefore, event-based control could be useful to deal with such situations. Specially, when control actions are expensive and the need to relax periodicity of computations and as a consequence to reduce the processor usage, is required. Event-based control is actually an alternative relying on aperiodic sampling where the inputs of the system are updated only when some events are generated.

Literature states contributions on event-based control, since the appearance of first papers [3], [36] and [4] up to some recent ones [34], [2], [15], [11], [28], [12] and [21] among others. We can find advantages of this strategy such as the fact that events are generated (different event algorithms are investigated) reducing the number sampling instants for the same final performance. Nevertheless, there exist few theoretical results about stability, convergence and performance.

One the other hand, many physical systems having an engineering interest are described by partial differential equations (PDEs). Actually, a large number of mathematical models in applied sciences are given by PDEs. Such systems are also called infinite-dimensional systems or distributed parameter systems due to fact that the state-space is infinite-dimensional or distributed. For a class of them, hyperbolic systems (both of conservation laws and of balance laws) stand out having important applications in modelling and control of physical networks. For example, hydraulic networks (shallow water (Saint-Venant) equations for open channels), road traffic networks (Aw-Rascle equations), gas pipeline networks (Euler equations for gas flow) are mostly considered in literature.

Concerning the control, several results are available for hyperbolic systems. For example, boundary control using backstepping [17] and [18]. Lyapunov techniques are used for the stability analysis; see for instance [6], [7], [10], [9]. Further contributions for stabilization of hyperbolic systems by means of Lyapunov techniques can be found in [26], [25], [35], [23] and [24]. The last reference is mainly the one we have studied. It will be followed throughout this document.

Due to the distributed nature of state space in hyperbolic systems, it is common for most of applications to have sensors and actuators available both distributed and in the boundary. Then, it is again a problem of networked control systems where the need to reduce communications and computational costs is a central issue. So, event-based control could be a good solution. However, event-based control theory for infinite-dimensional systems is not well developed. Therefore, this is a strong motivation to try to apply this new control strategy to such systems. This is the objective of our project and core of the document. We aim then at proposing an approach which combines what is already done of event-based control for finite-dimensional systems with classical boundary control (Lyapunov techniques) of linear hyperbolic systems. This is indeed an emerging field which is worth deeply researching.

This document is organized as follows: In Section 2, the basic ideas of event-based control issues are presented. Input-to.state-stability as well as control Lyapunov function are introduced so as to define two triggering conditions. One numerical example is illustrated for which a brief introduction of hybrid systems was carried out. In Section 3 we present our main contribution: by extending the notions given in Section 2, two triggering condition were defined. Several numerical examples are illustrated showing the validity of our results. Finally, conclusions and perspectives are given in Section 4.

2 Event-based control of finite dimensional systems

Event-based control, also called event-triggered control, contrary to traditional computer control systems where sampling is periodic, relies on aperiodic sampling. This sampling strategy updates the control value only when the system needs attention (i.e when an event occurs). For example, if we think in terms of trajectories (or solutions), the event algorithm updates the control only when they present significant changes or just when for instance, the state deviates more than a certain threshold from a desired value. Two elements are essential in this framework: the first one is the traditional feedback controller that computes the control input, and the second one, a trigger algorithm or mechanism which verifies a condition (called either triggering condition or event condition) such that when such a condition is violated, an event is triggered or generated.

Triggering conditions are namely related to stability-performance. Therefore, the use of Lyapunov techniques as well as input-to-state stability (ISS) issues are useful to define suitable triggering conditions.

In this section we introduce the main concepts of event-based control for finite-dimensional systems by mainly following [34], [12] and [21].

Let us consider a control system of the form:

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{2.1}$$

being $f : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz continuous on compacts, for which a feedback controller

$$u = k(x) \tag{2.2}$$

has been designed such that the closed loop system

$$\dot{x} = f(x, k(x + e)) \tag{2.3}$$

is input-to-state stable (ISS) with respect to measurements errors $e \in \mathbb{R}^n$. Further details about ISS notions and ISS-Lyapunov function can be found in Apendix B.1. For the sequel, a characterization ISS-Lyapunov function is needed.

Definition 1. *[34] A smooth function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ is said to be an ISS- Lyapunov function for the closed-loop system* (2.3) *if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, α and γ satisfying for all x, $e \in \mathbb{R}^n$*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$$
$$\nabla V(x) \cdot f(x, k(x + e)) \leq -\alpha(\|x\|) + \gamma(\|e|).$$

Typically, the implementation of the feedback law $u = k(x)$ on a digital platform is carried out by sampling the state at instants $t_0, t_1, t_2, ...$ so that the actual control input is given by $u(t_i) = k(x(t_i))$, $\forall t \in [t_i, t_{i+1}), i \in \mathbb{I}$. Thus, the sequence $\{t_i\}_{i\in\mathbb{I}}$ of sampling instants are the execution times at which the control input is computed and updated (between actuator updates, the input value u is held constant). Sequence of sampling instants $\{t_i\}_{i\in\mathbb{I}}$ is typically periodic, i.e $t_{i+1} - t_i = T$ where $T > 0$ is the period; but in our framework, it is no longer a periodic sequence. In addition, if there is an infinite number of executions, then $\mathbb{I} = \mathbb{N}$ [12].

Let us now define the measurement error e, associated with the i^{th} sampling time as $e : [t_i, t_{i+1}) \to \mathbb{R}^n$ in which

$$e(t) = x(t_i) - x(t), \quad \forall t \in [t_i, t_{i+1}) \tag{2.4}$$

The sampled data system controller uses $x(t_i) = e(t) + x(t)$ rather than $x(t)$. Then the sampled data system state must satisfy

$$\dot{x}(t) = f(x(t), k(x(t_i))) = f(x(t), k(x(t) + e(t))) \quad \forall t \in [t_i, t_{i+1})$$

which is indeed the closed-loop system (2.3) we want to deal with.

In event-based control, the execution times are triggered by events that are generated according to some execution rule given by a suitable condition. The set of triggering times $\{t_i\}_{i\in\mathbb{I}}$ can be formally defined by

$$t_0 = 0, \qquad t_{i+1} = \inf\{t > t_i| \;\text{ some execution rule}\}$$

One of the most important issues in event-based control is the existence of a *minimal inter-execution time* or *minimal inter-event time*, i.e some bound $\tau > 0$ such that the sequence $\{t_i\}_{i\in\mathbb{I}}$ satisfies

$$t_{i+1} - t_i \geq \tau \tag{2.5}$$

If the minimal inter-event time is zero, then an event-triggered implementation will require faster and faster updates and thus cannot be implemented on a digital platform [15]. This is precisely the so-called *Zeno phenomena.* The proof of the existence of such a $\tau > 0$ is given in [34] (see also [1] and [19] for further details).

In addition, it is defined in [21] an event function $ev : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that indicates if one needs ($ev \leq 0$) or not ($ev \geq 0$) to update the control value. Event function ev takes the current state $x(t)$ as input and a memory $m = x(t_i)$ of the state last time when it was necessary to update the control (i.e when an event happened). By linking with notation presented above, it is clear that $m = x(t_i) = e(t) + x(t)$. In that work, a universal formula for event-based stabilization of general nonlinear affine in the control is proposed, where the event function is related directly to the time derivative of a Control Lyapunov function. Zeno phenomena is avoided since the existence of the minimal inter-execution time is also proven.

Equipped with all previous preliminary notions, appropriate triggering conditions will be presented. This is the aim of the next two subsections.

2.1 ISS-based triggering condition

Under the ISS assumption aforementioned, we know that

$$\dot{V} \leq -\alpha(\|x\|) + \gamma(\|e\|)$$

If we *restrict* the error or actually the " gap " $\gamma(\|e\|)$ so that for some $\sigma \in (0,1)$

$$\gamma(\|e\|) \leq \sigma\alpha(\|x\|) \tag{2.6}$$

we get

$$\begin{aligned} \dot{V} \leq & -\alpha(\|x\|) + \sigma\alpha(\|x\|) \\ = & (\sigma - 1)\alpha(\|x\|) \end{aligned}$$

which guarantees that $x(t)$ converges asymptotically to the the origin. The level of performance can be adjusted using the parameter σ, in such a way that when σ approaches 0, the performance of the closed loop system approaches the system when $e(t) = 0$ for all $t \in \mathbb{R}_0^+$ [12].
Inequality (2.6) can be *enforced* by executing the control task when:

$$\gamma(\|e\|) \geq \sigma\alpha(\|x\|) \tag{2.7}$$

The triggering condition is therefore (2.7) and t_i as we stated previously becomes:

$$t_0 = 0, \quad t_{i+1} = \inf\{t \in \mathbb{R} | t > t_i \wedge \gamma(\|e\|) \geq \sigma\alpha(\|x\|)\}$$

The time evolution of both the threshold $\alpha(\|x\|)$ and the gap $\gamma(\|e\|)$ are shown in Fig. 1. The central idea can be explained as follows (see [19]): At the beginning of the interval $[t_i, t_{i+1})$, $e(t_i) = x(t_i) - x(t_i) = 0$. After that, the norm of the error increases. When condition (2.7) is satisfied, i.e $\gamma(\|e\|) > \sigma(\|\alpha\|)$, then the system state is again sampled, and therefore we enforce the error to zero again. It is important to emphasize that the event function defined by inequality (2.6) is continuously monitored. Actually, this is a particular feature of event-triggered control that differs of self-triggered control (subject which is beyond the scope of this document).

Finally, it is worth remarking that the decreasing of the Lyapunov function together with the existence of a minimal inter-execution time implies the asymptotic stability of the closed loop systems [12]. As a mater of fact, that study is the main contribution in [34].

The triggering condition that we have just presented, will be useful when extending the event-based control formulation to linear hyperbolic systems as we will deal with in Section 3.

2.2 $\dot{V}$-based triggering condition

Based on the work in [21], we want to particularly point out that the event function is given by

$$ev(x, m) = -\dot{V}_1 + \tilde{\sigma}\dot{V}_2 \tag{2.8}$$

for some $\tilde{\sigma} \in [0,1)$. $\dot{V}_1$ is the value of the time derivative of a control Lyapunov function ($\dot{V}$) when applying $u = k(m)$. $\dot{V}_2$ is the value of $\dot{V}$ if $u = k(x)$ is applied instead of $k(m)$. The triggering condition can be seen in the following:

$$t_0 = 0, \quad t_{i+1} = \inf\{t \in \mathbb{R} | t > t_i \wedge \dot{V}_1 \geq \tilde{\sigma}\dot{V}_2\}$$

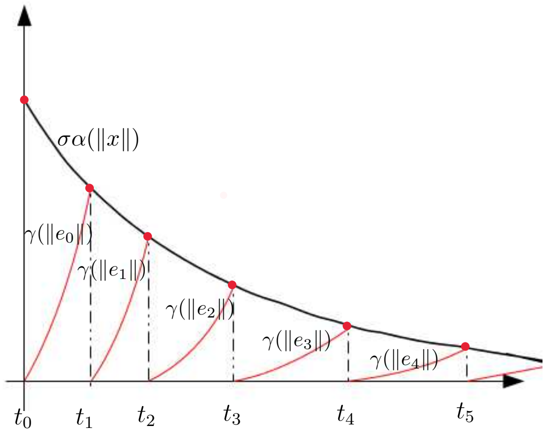


Figure 1: Trajectories of the gap and threshold

In order to understand the concept and where the idea comes from, let us present the subsequent simple analysis which does need ISS assumptions. Also, an numerical example illustrating the idea will be provided in next subsection.

Let us consider a linear system

$$\dot{x} = Ax + Bu \tag{2.9}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. A feedback controller $u = Kx$ is proposed rendering the closed-loop system globally asymptotically stable. This implies the existence of a Lyapunov function $V(x) = x^T Px$ where P is a symmetric positive definite matrix such that

$$\dot{V}(x) = \frac{\partial V}{\partial x}(A + BK)x = -x^T Qx \tag{2.10}$$

where Q is a symmetric positive matrix. Hence, V decreases, and the rate at which it decreases is specified by matrix Q. As it is stated in [15], if we are willing to tolerate a slower rate of decrease, we would require the solution of an event-based implementation to satisfy the *weaker* inequality

$$\dot{V}(x(t)) \leq -\tilde{\sigma} x^T(t)Qx(t) \tag{2.11}$$

for some $\tilde{\sigma} \in [0, 1)$. The requirement (2.11) suggests that we only need to recompute $u = Kx$ and update the actuator when (2.11) is violated, i.e. $\dot{V}(x) \geq -\tilde{\sigma} x^T Qx$.

Now, by introducing again the error e defined by $e(t) = x(t_i) - x(t)$ $\forall t \in [t_i, t_{i+1})$, we know that the evolution of the closed-loop system during the interval $[t_i, t_{i+1})$ is

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BKx(t_i) \\ &= Ax(t) + BK(x(t) + e(t))\end{aligned}$$

therefore, rewriting the time derivative of V yields the following:

$$\begin{aligned}\dot{V}(x(t)) &= \frac{\partial V}{\partial x}(A+BK)x(t) + \frac{\partial V}{\partial x}BKe(t)\\ &= -x^T(t)Qx(t) + 2x^T(t)PBKe(t)\end{aligned} \tag{2.12}$$

Now, let us specifically remark that the above expression (2.12) is the value of the time derivative when applying $u = k(x+e) = km$. In that sense, in (2.12), $\dot{V}_1 = \dot{V} = -x^T(t)Qx(t)+2x^T(t)PBKe(t)$. In addition, expression (2.10) is the value of the time derivative when applying $u = k(x)$. Then, $\dot{V}_2 = -x^TQx$.

Accordingly, substituting $\dot{V}_1 = \dot{V}$ in inequality (2.11), we have

$$\dot{V}_1 \leq -\tilde{\sigma}x^TQx = \tilde{\sigma}\dot{V}_2 \tag{2.13}$$

Hence, when condition (2.13) is violated, *an event is triggered*. The event function is then formalized as $ev(x,m) = -\dot{V}_1 + \tilde{\sigma}\dot{V}_2$.

This event function will also be useful when extending the event-based control formulation to linear hyperbolic systems. We will take advantage of Lyapunov techniques to get the candidate Lyapunov function and in turn its time derivative.

2.3 Hybrid system formulation and numerical example

Since event-based control systems can be regarded as a particular case of hybrid control systems, then by means of the Matlab toolbox HyEQ presented in "A Toolbox for Simulation of Hybrid Dynamical Systems" [27], some simulations were done. By following the framework used in [13] and [27], let us briefly introduce what an hybrid system is about: A hybrid system is a dynamical system with continuous and discrete dynamics. A state can both *flow* and *jump*. A hybrid system $\mathcal{H}$ on a state space $\mathbb{R}^n$ with input space $\mathbb{R}^m$ is then defined by the following objects:

- A set $C \subset \mathbb{R}^n \times \mathbb{R}^m$ called the flow set
- A function $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ called the flow map.
- A set $D \subset \mathbb{R}^n \times \mathbb{R}^m$ called the jumps set
- A function $g : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ called the jump map.

Hybrid system are given by hybrid equations:

$$\mathcal{H}: \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \quad \begin{cases} \dot{x} = f(x,u) & (x,u) \in C \\ x^+ = g(x,u) & (x,u) \in D \end{cases} \tag{2.14}$$

Solutions are functions parametrized by hybrid time (t,j). The flow map f defines the continuous dynamics on the flow set C, whereas the jump map g defines the discrete dynamics on the jump set. The core of the toolbox is the *HyEQsolver* solver. The flows are calculated using the built-in ODE solver function in Matlab. If the solution leaves the flow set C, the discrete event is detected using the function *zeroevents*. Once the state jumps, the next value of the state is calculated via the jump map g using the function jump.

To be more familiar with hybrid formulation, and specially, in order to implement a numerical example into HyEQ toolbox, we have also considered some issues condensed in [28];

The closed-loop system (2.1) in hybrid formulation [28] is

$$\begin{cases} \dot{x} = f(x,s) \\ \dot{s} = 0 & (x,s) \in \mathcal{F} \end{cases} \\ \begin{cases} x^{+} = x \\ s^{+} = u(x) = Kx & (x,s) \in \mathcal{J} \end{cases} \tag{2.15}$$

where $\mathcal{F}$ and $\mathcal{J}$ are the flow set and jump set respectively. $s \in \mathbb{R}^m$ represents the held value of the control input. More precisely, $s = Km$ where $m = x + e$.

In particular, according to triggering conditions previously defined, let us define $\mathcal{F}_1$ and $\mathcal{J}_1$ for the one in Subsection 2.1 and $\mathcal{F}_2$ and $\mathcal{J}_2$ for the another one in 2.2 as follows:

$$\mathcal{F}_1 = \{(x, s(e)) : \gamma(\|e\|) \le \sigma\alpha(\|x\|)\} \qquad \mathcal{F}_2 = \left\{(x,s) : \dot{V}_1 \le \tilde{\sigma}\dot{V}_2\right\}$$

$$\mathcal{J}_1 = \{(x, s(e)) : \gamma(\|e\|) \ge \sigma\alpha(\|x\|)\} \qquad \mathcal{J}_2 = \left\{(x,s) : \dot{V}_1 \ge \tilde{\sigma}\dot{V}_2\right\}$$

With the aim at implementing an example, we will just consider $\mathcal{F}_2 = \left\{(x,s) : \dot{V}_1 \le \tilde{\sigma}\dot{V}_2\right\}$ and $\mathcal{J}_2 = \left\{(x,s) : \dot{V}_1 \ge \tilde{\sigma}\dot{V}_2\right\}$ for the linear system (2.9) with $u = kx$. In this case rewriting the closed-loop system in a hybrid framework yields:

$$\begin{cases} \dot{x} = Ax + Bs \\ \dot{s} = 0 & (x,s) \in \mathcal{F}_2 \end{cases} \\ \begin{cases} x^{+} = x \\ s^{+} = u(x) = Kx & (x,s) \in \mathcal{J}_2 \end{cases} \tag{2.16}$$

Now, from, (2.12) we have

$$\begin{aligned} \dot{V}_1 = -x^TQx + 2x^TPBKe &= -x^TQx + 2x^TPBs - 2x^TBKx \\ &= -x^TQx + s^TB^TPx + x^TPBs - 2x^TPBKx \end{aligned}$$

$$\Leftrightarrow \dot{V}_1 = \begin{bmatrix} x \\ s \end{bmatrix}^T \begin{bmatrix} -Q - 2PBK & PB \\ B^TP & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} \tag{2.17}$$

and from (2.10) we have

$$\dot{V}_2 = -x^TQx$$

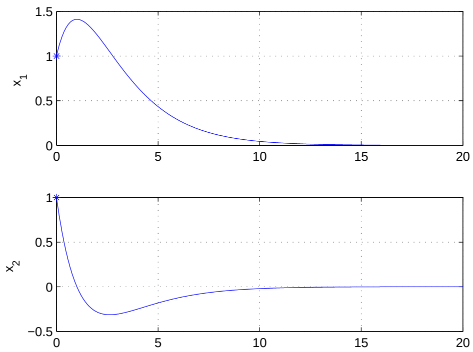
$$\Leftrightarrow \dot{V}_2 = \begin{bmatrix} x \\ s \end{bmatrix}^T \begin{bmatrix} -Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} \tag{2.18}$$

Hence, the event function which is considered when implementing in the toolbox is:

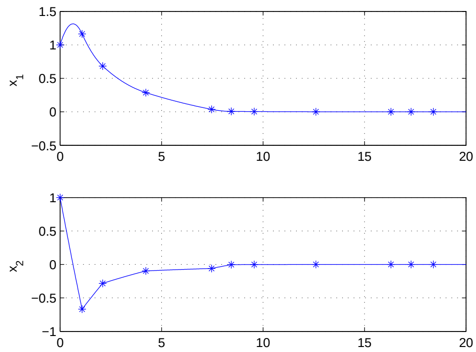
$$ev = -\begin{bmatrix} x \\ s \end{bmatrix}^T \begin{bmatrix} -Q - 2PBK & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} + \tilde{\sigma} \begin{bmatrix} x \\ s \end{bmatrix}^T \begin{bmatrix} -Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} \tag{2.19}$$

Example

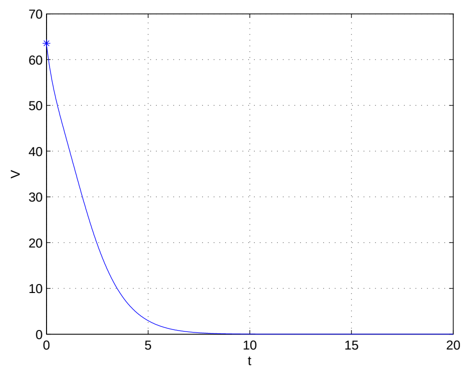
Consider the linear system in [28]. $\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} u$ stabilized by the following linear feedback $u = \begin{bmatrix} 3.75 & 11.75 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Using $V = x^T P x$ as a Lyapunov function with P defined by $P = \begin{bmatrix} 21.213 & 10.843 \\ 10.843 & 20.666 \end{bmatrix}$. Fig. 2 shows the simulation result of the closed-loop system using the continuous-time control and event-based control with the above $\dot{V}$-triggering condition getting 12 execution times.



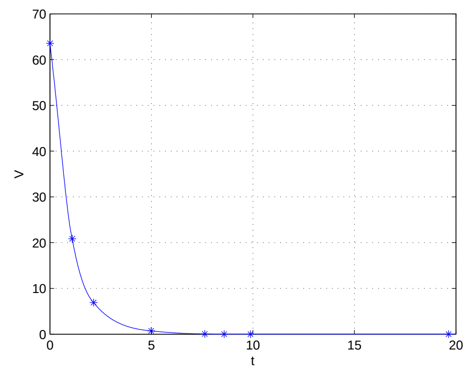
(a) Time evolution of states - continuous control



(b) Time evolution of states - event-based control



(c) Lyapunov function - continuous control



(d) Lyapunov function -event based control

Figure 2: Time evolution of states and Lyapunov functions.

3 Event-based control of linear hyperbolic systems

In this section, it is intended to extend the trigerring conditions presented in Sections 2.1 and 2.2 for finite dimensional systems but now for linear hyperbolic systems.

3.1 Linear hyperbolic systems

Linear hyperbolic systems of both conservation laws[1] and of balance laws (linear source term) are considered. The sequel is based on [24]:

We will mainly deal with linear hyperbolic systems of the form[2],

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = F y(t,x) \qquad x \in [0,1], t \in \mathbb{R}_+ \tag{3.1}$$

where $y : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^n$, F is a matrix in $\mathbb{R}^{n\times n}$, Λ is a diagonal matrix in $\mathbb{R}^{n\times n}$ such that $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ with $\lambda_k < 0$ for $k \in \{1, ..., m\}$ and $\lambda_k > 0$ for $k \in \{m+1, ..., n\}$. We use the notation $y = \begin{pmatrix} y^- \\ y^+ \end{pmatrix}$ where $y^- : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^m$ and $y^+ : \mathbb{R}_+ \times [0,1] \to \mathbb{R}^{n-m}$. We consider also the following boundary condition:

$$\begin{pmatrix} y^-(t,1) \\ y^+(t,0) \end{pmatrix} = G \begin{pmatrix} y^-(t,0) \\ y^+(t,1) \end{pmatrix} \qquad t \in \mathbb{R}_+ \tag{3.2}$$

where G is a matrix in $\mathbb{R}^{n\times n}$, made up of matrices $G_{--} \in \mathbb{R}^{m\times m}$, $G_{-+} \in \mathbb{R}^{m\times(n-m)}$, $G_{+-} \in \mathbb{R}^{(n-m)\times m}$ and $G_{++} \in \mathbb{R}^{(n-m)\times(n-m)}$ such that $G = \begin{pmatrix} G_{--} & G_{-+} \\ G_{+-} & G_{++} \end{pmatrix}$. System (3.1) may be graphically represented as a feedback control system as shown in Fig. 3.

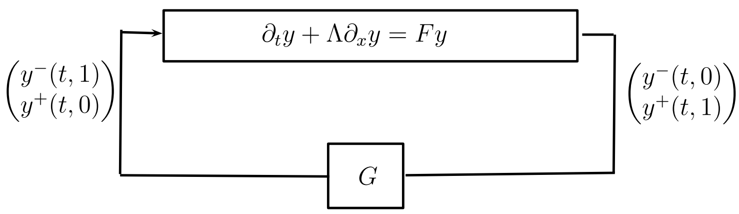


Figure 3: The hyperbolic system (3.1) viewed as a feedback control system.

In addition, we consider the initial condition given by

$$y(0,x) = y^0(x), \qquad x \in (0,1) \tag{3.3}$$

[1]see Appendix A.1 for a basic definition of a linear hyperbolic system of conservation laws.
[2]System given in Riemman coordinates.

where $y^0 \in L^2((0,1);\mathbb{R}^n)$.

It can be shown that there exists a unique solution $y \in C^0(\mathbb{R}_+; L^2((0,1);\mathbb{R}^n))$ to the initial value problem. Since solutions may not be differentiable everywhere, notion of weak solutions (generalized ones) of partial differential equations has to be used. In this document, we will not enter in details about that. In [9], for instance, more details are given.

Definition 2. *The linear hyperbolic system* (3.1)-(3.3) *is globally exponentially stable (GES) if there exist $\nu > 0$ and $C > 0$ such that, for every $y^0 \in L^2((0,1);\mathbb{R}^n)$, the solution of the initial (Cauchy) value problem* (3.1)-(3.3) *satisfies*

$$\|y(t,\cdot)\|_{L^2((0,1);\mathbb{R}^n)} \leq Ce^{-\nu t}\|y^0\|_{L^2((0,1);\mathbb{R}^n)} \quad \forall t \in \mathbb{R}_+ \tag{3.4}$$

Sufficient condition on the boundary conditions for exponential stability [7]

It is stated in [7] a sufficient condition, usually called *dissipative boundary condition* which guarantees exponential stability for the System (3.1) with F sufficiently small. That sufficient condition is therefore,

$$\rho_1(G) = \mathrm{Inf}\left\{\|\Delta G \Delta^{-1}\|; \Delta \in \mathcal{D}_{n,+}\right\} < 1 \tag{3.5}$$

where $\|\|$ denotes the usual 2-norm of matrices in $\mathbb{R}^{n\times n}$ and $\mathcal{D}_{n,+}$ denotes the set of diagonal matrices whose elements on the diagonal are strictly positive.

A general sufficient conditions for exponential stability [24]

The following result will be useful for the subsequent work when defining event-triggering conditions for linear hyperbolic systems of balance laws.

Proposition 1. *Let us assume that there exist $\nu > 0$, $\mu \in \mathbb{R}$ and symmetric positive definite matrices $Q^- \in \mathbb{R}^{m\times m}$ and $Q^+ \in \mathbb{R}^{(n-m)\times(n-m)}$ such that, defining for each $x \in [0,1]$, $\mathcal{Q}(x) = diag[e^{2\mu x}Q^-, e^{-2\mu x}Q^+]$, $\mathcal{Q}(x)\Lambda = \Lambda\mathcal{Q}(x)$, the following matrix inequalities hold*

$$-2\mu\mathcal{Q}(x)\Lambda^+ + F^T\mathcal{Q}(x) + \mathcal{Q}(x)F \leq -2\nu\mathcal{Q}(x) \tag{3.6}$$

$$\begin{pmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{pmatrix}^T \mathcal{Q}(0)\Lambda \begin{pmatrix} I_m & O_{m,n-m} \\ G_{+-} & G_{++} \end{pmatrix} \leq \begin{pmatrix} G_{--} & G_{-+} \\ O_{n-m,m} & I_{n-m} \end{pmatrix}^T \mathcal{Q}(1)\Lambda \begin{pmatrix} G_{--} & G_{-+} \\ O_{n-m,m} & I_{n-m} \end{pmatrix} \tag{3.7}$$

Then, there exists C such that (3.4) *and the linear hyperbolic system* (3.1)-(3.2) *is GES.*

The details of the proof are given in [24] and they are the basis on which we shall prove in Proposition 2, but roughly the idea behind the proof consist of taking as candidate Lyapunov function (based on the L^2-norm) for all $y \in L^2((0,1),\mathbb{R}^n)$

$$V(y) = \int_0^1 y(x)^T\mathcal{Q}(x)y(x)dx \tag{3.8}$$

and, as usual in Lyapunov techniques, computing its time-derivative along the solutions of (3.1)-(3.2). Once we get it, conditions are naturally imposed in order to obtain a strict Lyapunov condition.

3.2 Main contribution: Towards a definition of an event-triggering condition

We consider now the linear hyperbolic system (3.1) but with the following boundary conditions with disturbance $d(t) = \begin{pmatrix} d^- \\ d^+ \end{pmatrix}$, that is

$$\begin{pmatrix} y^-(t,1) \\ y^+(t,0) \end{pmatrix} = G \begin{pmatrix} y^-(t,0) \\ y^+(t,1) \end{pmatrix} + d(t) \quad t \in \mathbb{R}_+ \tag{3.9}$$

where $d^- : \mathbb{R}_+ \to \mathbb{R}^m$ and $d^+ : \mathbb{R}_+ \to \mathbb{R}^{n-m}$. It is important to emphasize that d will be related to the measurement errors as we proceeded in Section 2. In addition, we assume that d is bounded. Therefore, we will seek for input-to-state stability considerations in order to establish a triggering condition; indeed, since we deal with disturbance, the non-positivity of $\dot{V}$ cannot be guaranteed all time. Therefore, input-to-state stability considerations are still needed. This is the aim of the following proposition.

Proposition 2 (ISS- strict Lyapunov function)**.** *Under assumptions of Proposition 1, let $V(y)$ be given by* (3.8). *Then, along the trajectories of* (3.1) *with boundary conditions* (3.9), *it holds*

$$\dot{V}(y) \leq -\nu V + \lambda \|d\|^2$$

Proof. It can be verified that $\mathcal{Q}(x)\Lambda$ is symmetric, $\partial_x \mathcal{Q}(x)\Lambda = -2\mu \mathcal{Q}(x)\Lambda^+$ and $\partial_x(y^T \mathcal{Q}(x)\Lambda y) = y^T \mathcal{Q}(x)\Lambda \partial_x y + \partial_x y^T \mathcal{Q}(x)\Lambda y - 2\mu y^T \mathcal{Q}(x)\Lambda^+ y$ which will be useful for the proof.

Let us consider the Lyapunov function given by (3.8). By computing the time derivative of V along the solutions of (3.1) yields the following:

$$\dot{V}(y) = \int_0^1 \partial_t (y^T \mathcal{Q}(x) y) dx = \int_0^1 (y^T \mathcal{Q}(x) \partial_t y + \partial_t y^T \mathcal{Q}(x) y) dx$$

It can be clearly noticed from the definition of system that $\partial_t y = Fy - \Lambda \partial_x y$ and of course $\partial_t y^T = F y^T - \Lambda \partial_x y^T$. Hence,

$$\begin{aligned}
\dot{V}(y) =& \int_0^1 (y^T \mathcal{Q}(x)(Fy - \Lambda\partial_x y) + (Fy^T - \Lambda \partial_x y^T)\mathcal{Q}(x) y) dx \\
=& \int_0^1 -y^T \mathcal{Q}(x)\Lambda \partial_x y - \partial_x y^T \Lambda \mathcal{Q}(x) y dx + \int_0^1 y^T (F^T \mathcal{Q}(x) + \mathcal{Q}(x) F) y dx \\
=& \int_0^1 - \left[\partial_x (y^T \mathcal{Q}(x)\Lambda y) + 2\mu y^T \mathcal{Q}(x)\Lambda^+ y\right] dx + \int_0^1 y^T (F^T \mathcal{Q}(x) + \mathcal{Q}(x) F) y dx \\
=& - \left[y^T \mathcal{Q}(x)\Lambda y\right]_0^1 + \int_0^1 y^T (-2\mu \mathcal{Q}(x)\Lambda^+ + F^T \mathcal{Q}(x) + \mathcal{Q}(x) F) y dx \\
=& y^T(t,0)\mathcal{Q}(0)\Lambda y(t,0) - y^T(t,1)\mathcal{Q}(1)\Lambda y(t,1) \\
& + \int_0^1 y^T (-2\mu \Lambda^+ \mathcal{Q}(x) + F^T \mathcal{Q}(x) + \mathcal{Q}(x) F) y dx
\end{aligned}$$

$$\dot{V}(y) = \begin{pmatrix} y^-(t,0) \\ y^+(t,0) \end{pmatrix}^T \mathcal{Q}(0)\Lambda \begin{pmatrix} y^-(t,0) \\ y^+(t,0) \end{pmatrix} - \begin{pmatrix} y^-(t,1) \\ y^+(t,1) \end{pmatrix}^T \mathcal{Q}(1)\Lambda \begin{pmatrix} y^-(t,1) \\ y^+(t,1) \end{pmatrix} + \int_0^1 y^T(-2\mu\Lambda^+\mathcal{Q}(x) + F^T\mathcal{Q}(x) + \mathcal{Q}(x)F)y dx$$

Here, according to the boundary conditions (3.9), i.e

$$y^-(t,1) = G_{--}y^-(t,0) + G_{-+}y^+(t,1) + d^-$$

$$y^+(t,0) = G_{+-}y^-(t,0) + G_{++}y^+(t,1) + d^+$$

we get,

$$\begin{aligned}\dot{V}(y) = &\begin{pmatrix} y^-(t,0) \\ G_{+-}y^-(t,0) + G_{++}y^+(t,1) + d^+ \end{pmatrix}^T \mathcal{Q}(0)\Lambda \begin{pmatrix} y^-(t,0) \\ G_{+-}y^-(t,0) + G_{++}y^+(t,1) + d^+ \end{pmatrix} \\ &- \begin{pmatrix} G_{--}y^-(t,0) + G_{-+}y^+(t,1) + d^- \\ y^+(t,1) \end{pmatrix}^T \mathcal{Q}(1)\Lambda \begin{pmatrix} G_{--}y^-(t,0) + G_{-+}y^+(t,1) + d^- \\ y^+(t,1) \end{pmatrix} \\ &+ \int_0^1 y^T(-2\mu\Lambda^+\mathcal{Q}(x) + F^T\mathcal{Q}(x) + \mathcal{Q}(x)F)y dx\end{aligned} \tag{3.10}$$

Re-organizing a bit more,

$$\begin{aligned}\dot{V}(y) = &\left(\underbrace{\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}\begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}}_{z_1} + \underbrace{\begin{bmatrix} 0 \\ d^+ \end{bmatrix}}_{d_1}\right)^T \underbrace{\mathcal{Q}(0)\Lambda}_{D_1} \left(\underbrace{\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}\begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}}_{z_1} + \underbrace{\begin{bmatrix} 0 \\ d^+ \end{bmatrix}}_{d_1}\right) \\ &- \left(\underbrace{\begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}\begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}}_{z_2} + \underbrace{\begin{bmatrix} d^- \\ 0 \end{bmatrix}}_{d_2}\right)^T \underbrace{\mathcal{Q}(1)\Lambda}_{D_2} \left(\underbrace{\begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}\begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}}_{z_2} + \underbrace{\begin{bmatrix} d^- \\ 0 \end{bmatrix}}_{d_2}\right) \\ &+ \int_0^1 y^T(-2\mu\Lambda^+\mathcal{Q}(x) + F^T\mathcal{Q}(x) + \mathcal{Q}(x)F)y dx\end{aligned} \tag{3.11}$$

We will directly decouple the terms z_1 and d_1 as well as z_2 and d_2. We then get

$$\dot{V} = (z_1 + d_1)^T D_1(z_1 + d_1) - (z_2 + d_2)^T D_2(z_2 + d_2) - \nu V$$

where the last term was obtained by assumption (3.6). In addition, we know that

$$(z_1 + d_1)^T D_1(z_1 + d_1) = z_1^T D_1 z_1 + 2z_1^T D_1 d_1 + d_1^T D_1 d_1$$

and

$$
\begin{aligned}
0 \leq & \left(\frac{1}{\sqrt{\alpha}} z_1 - \sqrt{\alpha} d_1\right)^T D_1 \left(\frac{1}{\sqrt{\alpha}} z_1 - \sqrt{\alpha} d_1\right) \\
= & \frac{1}{\alpha} z_1^T D_1 z_1 - 2 z_1^T D_1 d_1 + \alpha d_1 T D_1 d_1
\end{aligned}
$$

$$
\Rightarrow 2 z_1^T D_1 d_1 \leq \frac{1}{\alpha} z_1^T D_1 z_1 + \alpha d_1^T D_1 d_1
$$

for all $\alpha > 0$, which is the *Young's inequality.* Accordingly,

$$
(z_1 + d_1)^T D_1 (z_1 + d_1) \leq (1 + \alpha) z_1^T D_1 z_1 + (1 + \frac{1}{\alpha}) d_1^T D_1 d_1
$$

Similarly $-2 z_2^T D_2 d_2 \leq \frac{1}{\beta} z_2^T D_2 z_2 + \beta d_2^T D_2 d_2$ for all $\beta \geq 0$, hence

$$
-(z_2 + d_2)^T D_2 (z_2 + d_2) \leq -(1 - \beta) z_2^T D_2 z_2 - (1 - \frac{1}{\beta}) d_2^T D_2 d_2
$$

Finally,

$$
\begin{aligned}
\dot{V} \leq & -\nu V + (1 + \alpha) z_1^T D_1 z_1 - (1 - \beta) z_2^T D_2 z_2 + (1 + \frac{1}{\alpha}) d_1^T D_1 d_1 - (1 - \frac{1}{\beta}) d_2^T D_2 d_2 \\
= & -\nu V + (1 + \alpha) z_1^T D_1 z_1 - (1 - \beta) z_2^T D_2 z_2 + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}^T \begin{pmatrix} (1 + \frac{1}{\alpha} D_1) & 0 \\ 0 & -(1 - \frac{1}{\beta}) D_2 \end{pmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
\end{aligned}
$$

By assuming both α and β small enough, for the second term, we use assumption (3.7) in such a way that

$$
\begin{aligned}
& \underbrace{\begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}^T \begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^T}_{z_1^T} \underbrace{\mathcal{Q}(0)\Lambda}_{D_1} \underbrace{\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix} \begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}}_{z_1} \\
- & \underbrace{\begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}^T \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^T}_{z_2^T} \underbrace{\mathcal{Q}(1)\Lambda}_{D_2} \underbrace{\begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix} \begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}}_{z_2} \leq 0
\end{aligned}
$$

Accordingly, we finally achieve an ISS-Lyapunov function,

$$
\dot{V} \leq -\nu V + \lambda \left\| \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right\|^2
$$

Actually, computing either $\left\| \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right\|^2$ or $\|d\|^2$ one gets the desired result:

$$\dot{V} \leq -\nu V + \lambda \|d\|^2 \tag{3.12}$$

where λ can be taken as the largest eigenvalue of matrix $\begin{pmatrix} (1+\frac{1}{\alpha})D_1 & 0 \\ 0 & -(1-\frac{1}{\beta})D_2 \end{pmatrix}$. $\square$

It can be noticed that in previous formulation, the disturbance was taken in the boundary. In Appendices either A.4 or B.2 it can be seen that a disturbance is considered as a distributed one i.e. in the dynamics.

3.3 ISS-based triggering condition for linear hyperbolic systems

By following exactly the same ideas as in Section 2.1 (see inequality (2.6)), from (3.12) we will *restrict* d to satisfy:

$$\lambda \|d\|^2 \leq \nu\sigma V \tag{3.13}$$

getting a sort of *weaker* inequality. In that case

$$\dot{V} \leq -\nu(1-\sigma)V \qquad \sigma \in (0,1)$$

therefore, when (3.13) is violated, *an event is triggered*. Or, similarly

$$\begin{aligned} \dot{V} &\leq -\nu V + \lambda\|d\|^2 \\ &= -\nu(1-\sigma)V - \nu\sigma V + \lambda\|d\|^2 \end{aligned}$$

such that

$$-\nu\sigma V + \lambda\|d\|^2 \leq 0 \tag{3.14}$$

So, when (3.14) is violated, i.e $\lambda\|d\|^2 \geq \nu\sigma V$, *an event is triggered*. Hence, t_i formally defined becomes:

$$t_0 = 0, \quad t_{i+1} = \inf\{t \in \mathbb{R} | t > t_i \wedge \|d\|^2 \geq \frac{\nu\sigma}{\lambda} V\} \tag{3.15}$$

The main idea was then to relate $d(t)$ to $e(t)$. Assuming $G = H + BK$, where both H and B are given by the model and K was supposed to be computed rendering the linear hyperbolic system (3.1)-(3.2) global exponentially stable (according to sufficient conditions for stability presented in previous subsection); thus

$$d(t) = BKe(t) \tag{3.16}$$

Let us recall that $e(t)$ is the measurement error (Equation (2.4) for finite-dimensional case). In this framework it becomes $e(t) = -\begin{pmatrix} y^-(t,0) \\ y^+(t,1) \end{pmatrix} + \begin{pmatrix} y^-(t_i,0) \\ y^+(t_i,1) \end{pmatrix}$.

Theorem 1. *Under the assumptions of Proposition 1, the system* (3.1) *with boundary conditions* (3.9) *and $d(t)$ given by* (3.16) *with the triggering condition* (3.15)*, is globally asymptotically stable.*

The proof is an immediate consequence of Proposition 2 where the Lyapunov function is given by (3.8).

3.4 $\dot{V}$-based triggering condition for linear hyperbolic systems

As we mentioned in Subsection 2.2, it is also possible to do an extension of triggering condition for finite-dimensional systems to hyperbolic systems by just considering the Lyapunov function. Indeed, what is interesting is that we have explicitly $\dot{V}(y)$ given by the Equation (3.11). Therefore, the event function is:

$$ev = -\dot{V}_1 + \tilde{\sigma}\dot{V}_2$$

where $\dot{V}_1$ is the value of the time derivative of a control Lyapunov function ($\dot{V}$) when applying $K\begin{pmatrix} y^-(t_i,0) \\ y^+(t_i,1) \end{pmatrix}$ last time it was necessary an updated . $\dot{V}_2$ is the value of $\dot{V}$ if $K\begin{pmatrix} y^-(t,0) \\ y^+(t,1) \end{pmatrix}$ is applied instead. The triggering condition for the linear hyperbolic system under consideration can be seen in the following:

$$t_0 = 0, \quad t_{i+1} = \inf\{t \in \mathbb{R} | t > t_i \wedge \dot{V}_1 \geq \tilde{\sigma}\dot{V}_2\} \tag{3.17}$$

Theorem 2. *Under the assumptions of Proposition 1, the system* (3.1) *with boundary conditions* (3.9) *and $d(t)$ given by* (3.16) *with the triggering condition* (3.17)*, is globally asymptotically stable.*

The proof is an immediate consequence of Proposition 2 where the time derivative of the Lyapunov function is given by (3.11).

3.5 Numerical Examples

Numerical simulations were done by discretizing the linear hyperbolic system. For that purpose we have used a two-step variant of the Lax–Friedrichs (LxF) numerical method presented in [30] and the solver on Matlab in [29]. We select the parameters of the numerical scheme so that the *Courant-Friedrich-Levy* (CFL) condition for the stability holds. (For further details about the numerical method see Appendix A.3).

In addition, the sufficient stability conditions we have been dealing with throughout the section, i.e both (3.5) and those matrix inequalities presented in Proposition 1, were solved using classical numerical tools. For instance, Condition (3.5) can be solved using semi-definite programming (see e.g, Yalmip toolbox [20] with SeDuMi solver). For Conditions (3.6)-(3.7), semi-definite programming combining with the line search algorithm were used. The implementation was carried out in CVX toolbox, a package for specifying and solving convex programs [14].

On the other hand, regarding the implementation of the event-triggering condition for hyperbolic systems, the ideas for that purpose were based on the analysis of hybrid systems such as in the case of finite-dimensional systems presented in Subsection 2.3. Accordingly, we have followed the programming logic presented in [27] (HyEQ -toolbox); specially while thinking in flow set and jump set to be able to verify a triggering condition. And of course, we worked on the toolbox from [29] by just doing some modifications and taking into account the following facts:

- Two new functions are created: *flow set (InsideC)* and *jump set (InsideD)* are defined. Let us recall that the event function related to the triggering condition is put in the jump set function.

- In finite-dimensional systems we dealt with Ordinary differential equations (ODEs) which can be solved via the solver function ODE45 of MATLAB after having set the options for the solver : *options = odeset(options,'Events',(t,x) zeroevents(x,C,D,rule))* , for instance. In general, in order to stop automatically the integration, an *event location* is defined into the function called *Events*. Just there, the triggering condition, or furthermore, the event function is put. Once the integration is stopped, the algorithm updates the value of the controller and re-starts the integration with the new initial conditions.

- However, while solving partial differential equations, such a function does not exist. That tool is not available for the toolbox that we have chosen to solve PDEs .

- Let us mention rapidly that the PDE solver (see again [27] for more details) integrates the solution via *sol = hpde(sol,howfar,timestep)* after having defined the problem to be solved : *sol = setup(form,pdefun,t,x,u,method,periodic,bcfun,Neumann)*. Moreover, boundary conditions are defined as follows *[uL,uR] = bcfun(t,uLex,uRex)* which allows us to define explicitly the boundary conditions with disturbance by just adding the disturbance term, i.e. *[uL,uR] = bcfun(t,uLex,uRex) + d*. That was really an advantage of having worked with such a toolbox.

- Therefore, to stop integrating the solution of the PDEs, we have just not to call the function *sol = hpde(sol,howfar,timestep)*. In other words, to go out of the loop when an event occurs. Once the integration has stopped, the algorithm updates the value of the controller and re-stars the integration with new initial condition (last values) by calling again the function *hpde*.

- We have considered the particular case when $m = 0$ and $n = 2$. It simplifies a lot of things in our formulation just for a numerical simulation tractability. So, we can briefly summarize as follows: The hyperbolic system (3.1) is a 2×2 system of balance laws: $\partial_t y(t,x) + \Lambda \partial_x y(t,x) = Fy(t,x) \quad x \in [0,1], t \in \mathbb{R}_+$ where $y(t,x) \in \mathbb{R}^2$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2) \in \mathbb{R}^{2\times 2}$. $F \in \mathbb{R}^{2\times 2}$ eventually identical to zero matrix for a system of conservation laws. The boundary conditions (3.9) are now $y(t,0) = Gy(t,1) + d(t)$, where $G \in \mathbb{R}^{2\times 2}$ and $d \in \mathbb{R}^2$ eventually identical to zero in continuous control case. The Lyapunov (3.8) function is then $V(y) = \int_0^1 y(x)^T Q y(x) e^{-2\mu} dx$ and its time derivative:

$$\begin{aligned}\dot{V}(y) &= (Gy(t,1)+d)^T Q\Lambda\,(Gy(t,1)+d) - (y(t,1))^T e^{-2\mu} Q\Lambda\,(y(t,1)) \\ &+ \int_0^1 y^T(-2\mu\Lambda e^{-2\mu x} Q + F^T e^{-2\mu x} Q + e^{-2\mu x} QF) y dx\end{aligned}$$

The sufficient conditions (3.6) and (3.7) become then:

$$-2\mu Q\Lambda + F^T Q + QF \leq -2\nu Q$$

$$G^T Q\Lambda G < e^{-2\mu} Q\Lambda$$

We suppose also that eigenvalues of Λ are all positive or negative. In the following examples such a fact is evident.

The Event-trigger algorithm for linear hyperbolic systems is roughly presented in Algorithm 1 in Appendix A.5 . In that algorithm let us remark that *Inside* C means that the state flows in flow set. *Inside* D means that the state is into jump set, so it is said to be jumped due to an event occurs. e and d are global variables changing and being monitored all time. Event function defined according to a triggering condition is monitored all time as well. Lyapunov function is computed by approximative sums in a sub-function according to the particular case of formula (3.8).

Example 1: Continuous case

a. Linear hyperbolic system of conservation laws borrowed from [23] Section 4. As aforementioned, it is a particular case when $m = 0$ and $n = 2$ in (3.1) (3.2).

$$\partial_t y(t, x) + \Lambda\partial_x y(t, x) = 0 \qquad x \in [0, 1], t \in \mathbb{R}_+ \tag{3.18}$$

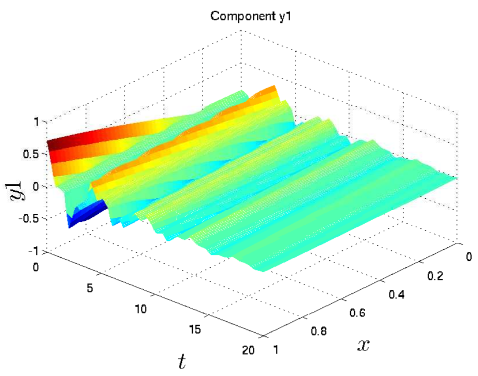
where $y(t, x) \in \mathbb{R}^2$. $\Lambda = \text{diag}(1, 1)$. The Boundary condition is then $y(0, t) = Gy(t, 1)$ where: $G = \begin{pmatrix} 0.1 & 0.6 \\ -1.2 & 0.1 \end{pmatrix}$ and initial conditions $y(0, x) = \begin{pmatrix} sin(x) \\ 0 \end{pmatrix}$. It was verified the condition (3.5) , that is $\|\Delta G\Delta^{-1}\| = 0.8769 < 1$ and thus $\rho_1(G) < 1$ with $\Delta = \begin{pmatrix} 1.4435 & 0 \\ 0 & 1.0477 \end{pmatrix}$. Besides this, sufficient conditions in proposition 1 were also checked getting as a result the existence of scalars $\mu = 0.1$, $\nu = 1.6021$ and one symmetric matrix $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0.5037 \end{pmatrix}$. Therefore, the hyperbolic system (3.18) with the related boundary condition is globally exponentially stable. This fact is strongly reinforced by the simulations in Fig. 4.

b. Now, let us consider a linear hyperbolic system of balance laws as follows:

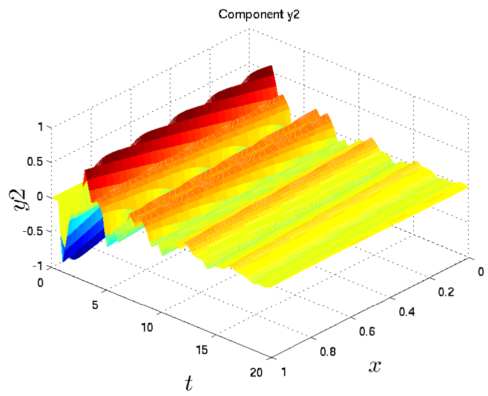
$$\partial_t y(t, x) + \Lambda\partial_x y(t, x) = Fy(t, x) \qquad x \in [0, 1], t \in \mathbb{R}_+ \tag{3.19}$$

where $y(t, x) \in \mathbb{R}^2$. $\Lambda = \text{diag}(1, 1)$. The Boundary condition is as before $y(0, t) = Gy(t, 1)$ but: $G = \begin{pmatrix} 0 & -1.2 \\ 0.6 & 0 \end{pmatrix}$ and initial conditions $y(0, x) = \begin{pmatrix} sin(x) \\ cos(x) \end{pmatrix}$. Here $F = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \end{pmatrix}$ is claimed to be small in terms of its norm (see next remark and thereafter Appendix A.4 to know more about this interesting fact).

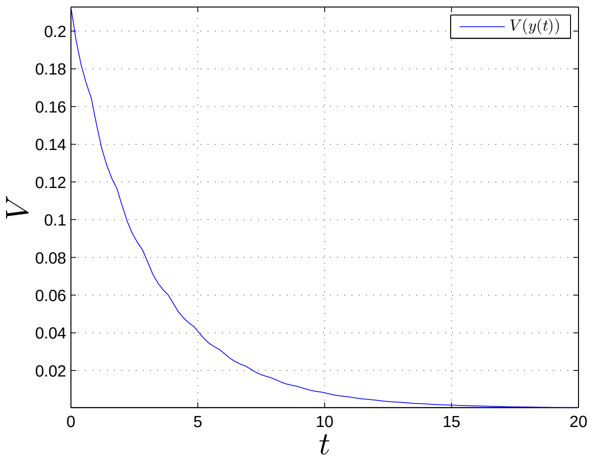
Again, it was verified the sufficient stability conditions, ending up $\|\Delta G\Delta^{-1}\| = 0.8769 < 1$ and thus $\rho_1(G) < 1$, with $\Delta = \begin{pmatrix} 1.0456 & 0 \\ 0 & 1.4373 \end{pmatrix}$. Also, $\mu = -0.2$, $\nu = 1.4092$ and



(a) Component y_1



(b) Component y_2

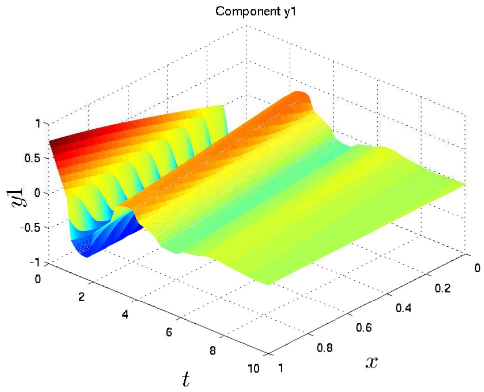


(c) Lyapunov function

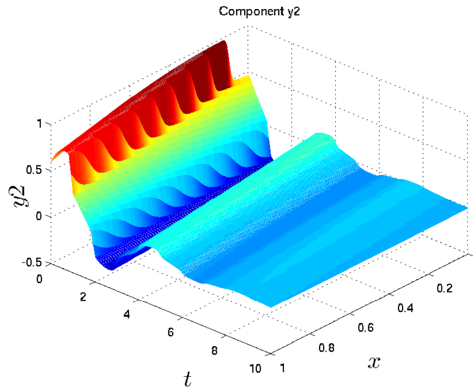
Figure 4: Time evolution of the first component y_1 (a) and of the second component y_2 (b) for the System (3.18) in Example 1-a. Also the time evolution of the Lyapunov function.

symmetric matrix $Q = \begin{pmatrix} 0.7466 & 0 \\ 0 & 1 \end{pmatrix}$. The hyperbolic system (3.18) with the related boundary condition is then globally exponentially stable. Fig. 5 shows the result of the simulations.

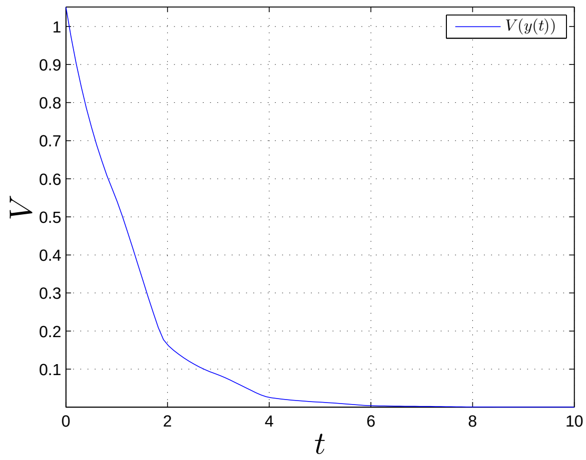
Remark: F is claimed to be small enough. This fact, is by the way, a sufficient condition for stability together with the ones presented before. In Appendix A.4 we illustrate examples by which it is possible to get stability or unstability. Besides that, we shall try, based on ISS issues for infinite-dimensional systems, to estimate an upper bound for the size of F.



(a) Component y_1



(b) Component y_2



(c) Lyapunov function

Figure 5: Time evolution of the first component y_1 (a) and of the second component y_2 (b) for the System (3.19) in Example 1-b. Also the time evolution of the Lyapunov function.

Example 2: Event-based case (ISS-based triggering condition)

a. Let us consider the linear hyperbolic system of balance laws:

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = F y(t,x) \qquad x \in [0,1], t \in \mathbb{R}_+ \tag{3.20}$$

where $y(t,x) \in \mathbb{R}^2$. $\Lambda = \text{diag}(1,1)$. The Boundary condition is again $y(0,t) = Gy(t,1)$ with: $G = \begin{pmatrix} 0 & -1.2 \\ 0.6 & 0 \end{pmatrix}$. $F = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \end{pmatrix}$, but initial conditions $y(0,x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have taken $\sigma = 0.3$ and $\alpha = 0.3$.

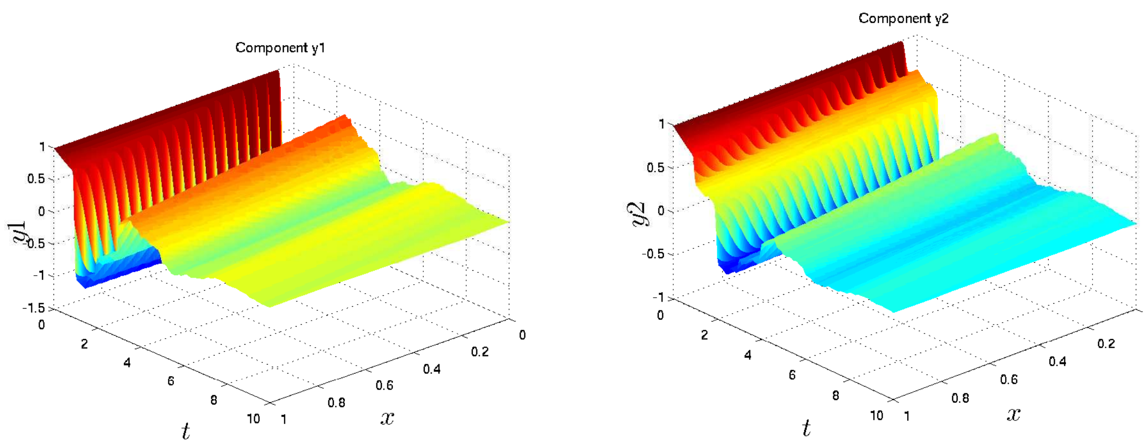
Figures 6-(a)-(b) show the result of simulation concerning the components of solution. With the parameters we have chosen, the trigger algorithm produced 54 execution times. It is considerably less with respect to continuous time since the time mesh points for all simulations into the numerical scheme is $NT = 200$. Fig. 7-(a) shows two decreasing

Lyapunov functions, the green line and the blue one with execution times for continuous case and for ISS-event case respectively. It can be noticed that while presenting less execution times as in the event-based case, the rate of convergence is slower.

In addition, Fig. 7-(b) shows the time evolution of trajectories $\frac{\nu\sigma V}{\lambda}$ and $\|d\|^2$. When trajectory $\|d\|^2$ exceeds $\frac{\nu\sigma V}{\lambda}$, meaning that the triggering condition is satisfied, an event occurs; and right after we enforce d to zero again. It can be also noticed in both Fig. 7-(b) and Fig. **??**-(c) (recall that event function is given by $ev = -\frac{\nu\sigma V}{\lambda} + \|d\|^2$) that there are some peaks. This is mainly due to the discontinuity of solution. Such a discontinuity propagates along the space and it appears at certain moments according to the entries values of matrix Λ. Moreover, the first huge peak appears at $t = 1$ and propagates according to values of Λ. We strongly believe that it because the initial conditions do not satisfy the so-called *zero-order compatibility condition*. In our case of study, $y(0,0) = Gy(0,1)$ must be satisfied (see e.g. [26] or [8] for further information. We will not enter in full details). Indeed,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 & -1.2 \\ 0.6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

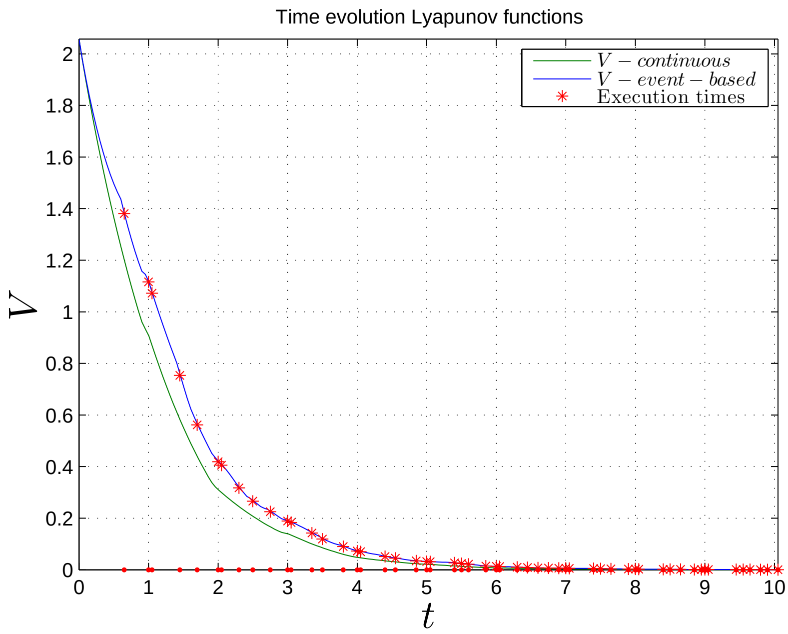
Nevertheless, such a huge peak could be avoided at $t = 1$ by suitably choosing initial conditions that satisfy the compatibility condition as we will illustrate in next example. It is important to empathize anyway that solution may be no continuous and $\dot{V}$ is negative all time, almost everywhere, excepting in a finite number of discontinuities.



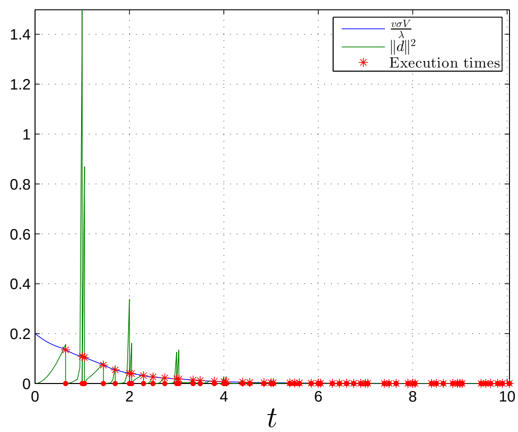
(a) Component y_1

(b) Component y_2

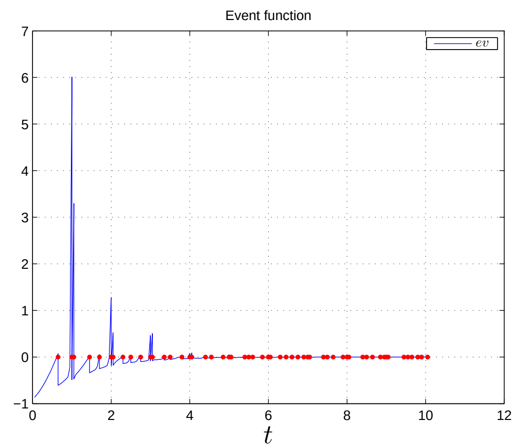
Figure 6: Time evolution of the first component y_1 (a) and of the second component y_2 for the System (3.20) in Example 2-a.



(a) Lyapunov functions - continuous and event-triggered cases



(b) Time evolution of trajectories $\frac{\nu\sigma V}{\lambda}$ and $\|d\|^2$



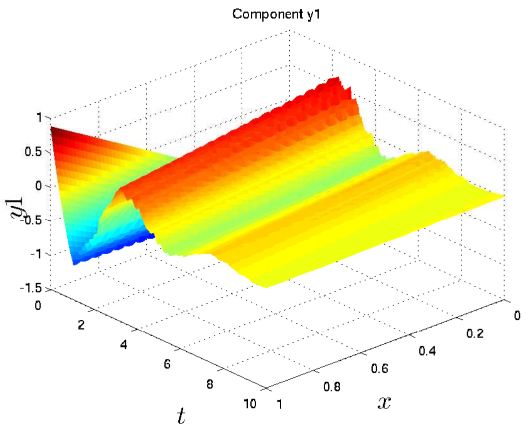
(c) Event function

Figure 7: In (a) Time evolution Lyapunov functions for the for the System (3.20) in Example 2-a. (b) Time evolution of the trajectories according to triggering condition (3.15) and (c) Event function.

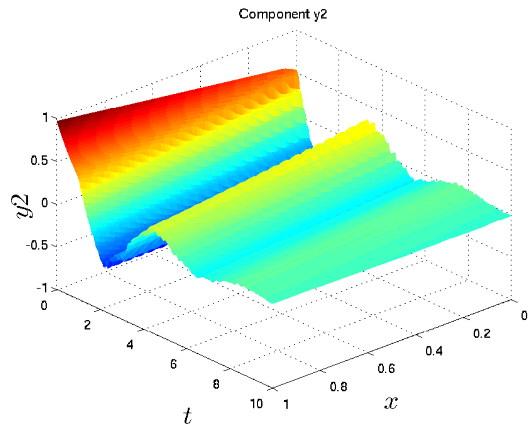
b. Now let us consider the same system of balance laws together with the same G as before but different initial conditions. For example, $y(0,x) = \begin{pmatrix} 2.2x - 1.2 \\ 0.4x + 0.6 \end{pmatrix}$. It can be rapidly verified that the zero-order compatibility condition $y(0,0) = Gy(0,1)$ is satisfied. In that sense, the first huge peak should not appear at $t = 1$. The triggering parameters for this example are: $\sigma = 0.65$ and $\alpha = 0.2$.

Figures 8-9 show the results of the simulation where we have got 50 events. Fig. 9-(a) shows the comparison between the Lyapunov functions for continuous case and event-base case. The blue line with execution times (red stars) seems to get away from the green line. Again, the convergence for event-based control is slower than in continuous case.

Furthermore, in Figures 9-(b)-(c), some peaks are evident. This behaviour is worth remarking because it depends directly on the solution and the regularity of it. This is one reason, we have not mentioned before, for which we have worked in L^2-norm. However, it can be also noticed that no huge peak appears at $t = 1$. This fact strongly justifies that the requirements that initial conditions should satisfy zero-order compatibility condition.

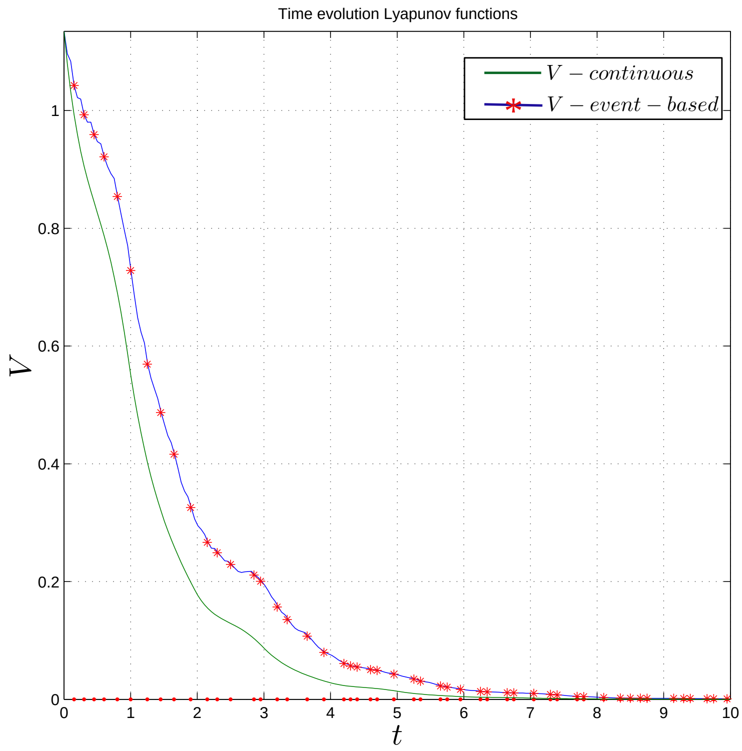


(a) Component y_1

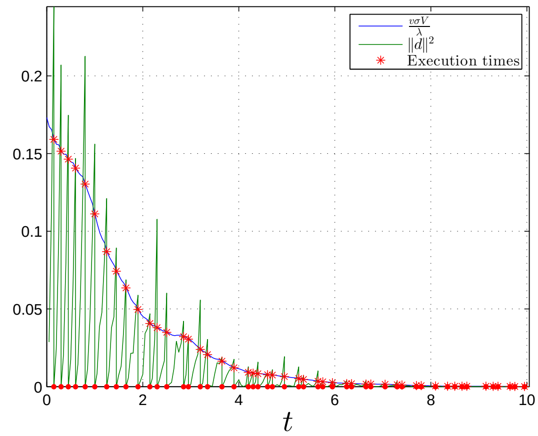


(b) Component y_2

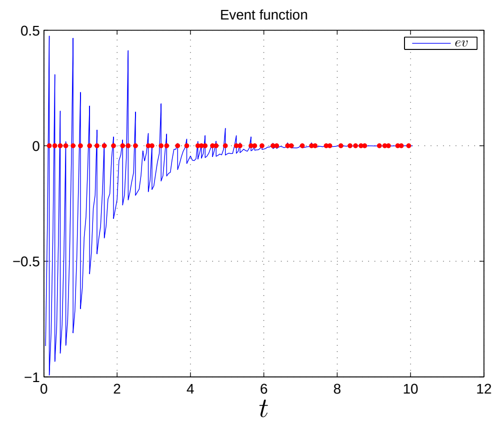
Figure 8: Time evolution of the first component y_1 (a) and of the second component y_2 for the system of balance laws in Example 2-b.



(a) Lyapunov functions- continuous and event-triggered cases



(b) Time evolution of trajectories $\frac{\nu\sigma V}{\lambda}$ and $\|d\|^2$



(c) Event function

Figure 9: (a) Lyapunov functions comparison for the system of balance laws in Example 2-b. (b) Time evolution of the trajectories according to triggering condition (3.15) and (c) Event function

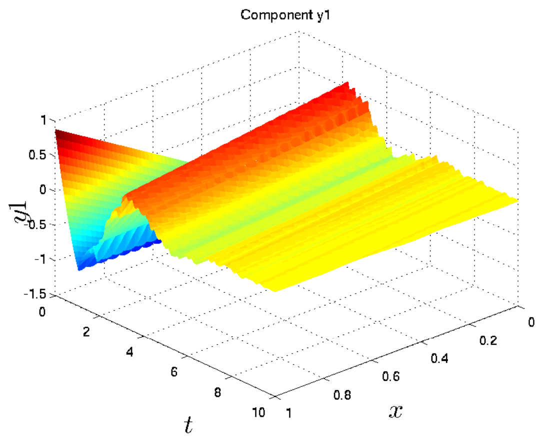
Example 3: Event-based case ($\dot{V}$-based triggering condition)

Let us consider exactly the same the linear hyperbolic system of balance laws as before:

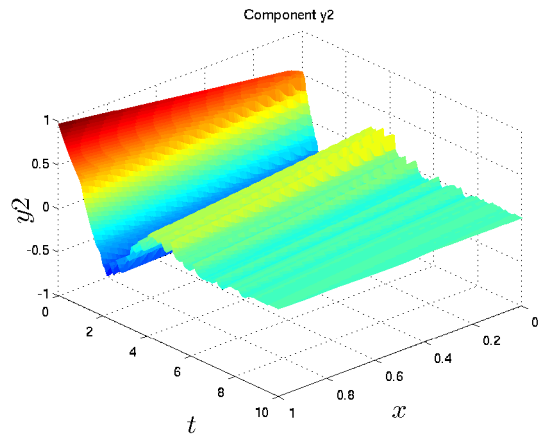
$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = F y(t,x) \qquad x \in [0,1], t \in \mathbb{R}_+ \tag{3.21}$$

where $y(t,x) \in \mathbb{R}^2$. $\Lambda = \mathrm{diag}(1,1)$. The Boundary condition is again $y(0,t) = Gy(t,1)$ with: $G = \begin{pmatrix} 0 & -1.2 \\ 0.6 & 0 \end{pmatrix}$. $F = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \end{pmatrix}$ and initial conditions $y(0,x) = \begin{pmatrix} 2.2x - 1.2 \\ 0.4x + 0.6 \end{pmatrix}$ satisfying the zero-order compatibility condition. In this example, we consider the triggering condition shown in (3.17) with $\tilde{\sigma} = 0.2$.

Figures 10-11 shows the result of simulations. The number of events at the end was 41. The event function shown in Fig 11-(b) differs from the ones presented before because the event function ev here in this approach, is positive between two execution times. Some peaks (negative values) are also presented showing again that they depend on the nature of solution and not on the trigger algorithm.

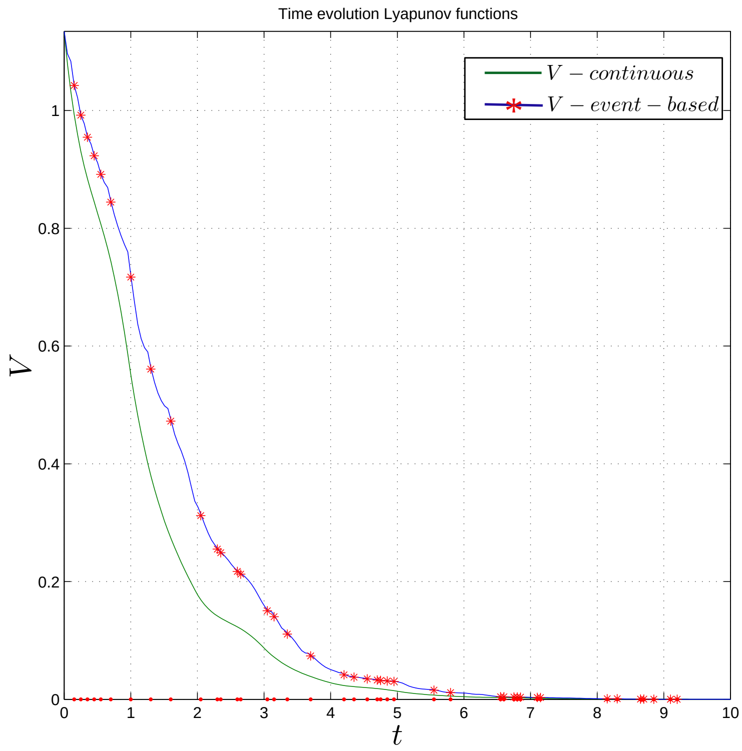


(a) Component y_1

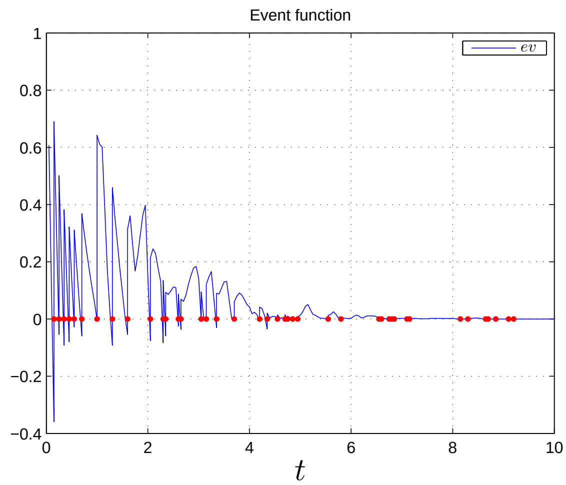


(b) Component y_2

Figure 10: Time evolution of the first component y_1 (a) and of the second component y_2 (b) for the System of balance laws (3.21) in Example 3.



(a) Lyapunov function



(b) Event function

Figure 11: In (a) Lyapunov functions for the System of balance laws (3.21) in Example 3. In (b) Event function according to triggering condition (3.17).

Example 4: Comparison between event-trigger strategies and concluding remarks

In order to see a global behaviour about the convergence and performance related to the number of events when varying triggering parameters, two tables were proposed. Table 3.1 the results for the ISS-based triggering condition approach. Table 3.2 instead, for the $\dot{V}-$based triggering condition approach.

Parameter σ	Number of events	Average inter-execution	$S = \int_0^1 V dt$
0.1	105	1.9048	1.5246
0.2	76	2.6316	1.6428
0.3	69	2.8986	1.6518
0.4	57	3.5088	1.8140
0.5	51	3.9216	1.8371
0.6	50	4	1.9081
0.7	47	4.2553	1.9880
0.8	43	4.6512	1.9718
0.9	45	4.4444	2.0132

Table 3.1: For ISS-based strategy

Parameter $\tilde{\sigma}$	Number of events	Average inter-execution	$S = \int_0^1 V dt$
0.1	34	5.8824	1.9628
0.2	41	4.8780	1.8357
0.3	38	5.2632	1.8435
0.4	52	3.8462	1.7405
0.5	62	3.2258	1.6790
0.6	70	2.8571	1.5783
0.7	89	2.2472	1.5223
0.8	120	1.6667	1.4511
0.9	156	1.2821	1.3976

Table 3.2: For $\dot{V}$-based strategy

For both tables, the area S under the Lyapunov graph and $t-$axis, increases as soon as the number of event decreases. It means that the rate of convergence gets slower. Indeed those area values are always less than $S = 1.3878$ which corresponds to the area for continuous case. In Table 3.1, for example, it can be noticed that when varying the parameter σ up to 0.9, we have got a less possible number of events (43) with $\sigma = 0.8$ but not with $\sigma = 0.9$ as intuitively expected. The third column of both tables is the average inter-execution times where can be obtained just by the quotient between the number of time mesh point and the number of events.

For the $\dot{V}$- based triggering strategy, it is reported in Table 3.2 the minimum number of events (34) has been obtained with $\tilde{\sigma} = 0.1$. The area S seems to be very closed to the one which was got for $\sigma = 0.9$ with 45 events in Table 3.1. Fig. 13 can reinforce the argument. There is a comparable performance between both approaches. It can be noticed also that in both strategies, the variation of S with respect to the number of events is not too marked.

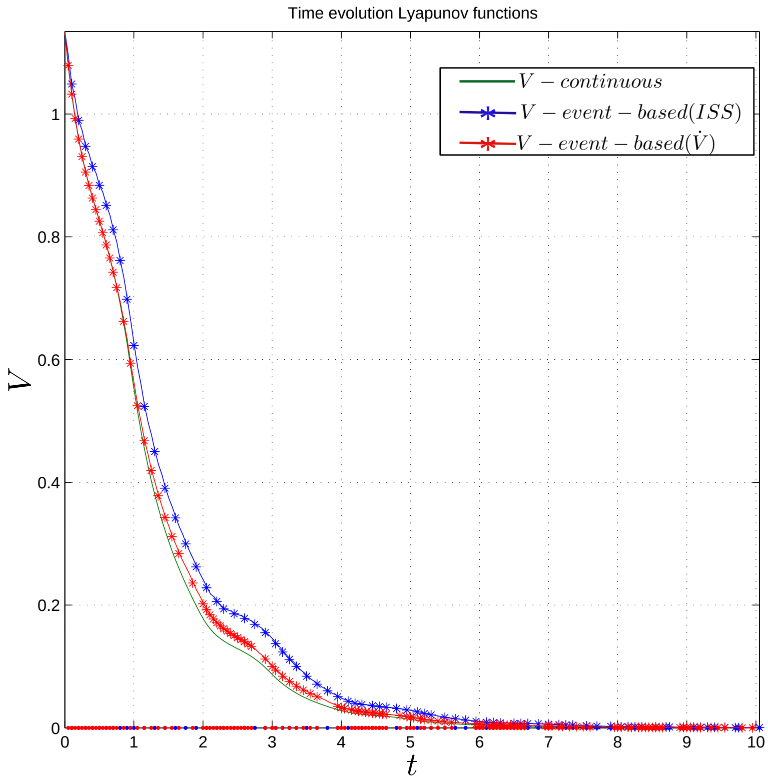


Figure 12: Lyapunov functions for both continuous and the two event-triggered cases ($\sigma = 0.2$ and $\tilde{\sigma} = 0.8$)

Fig. 12 and Fig. 13 show three Lyapunov functions for the System (3.21). The green line for continuous case and the red and blue ones for event-triggered case. The interesting thing to remark in both figures is that, Lyapunov functions for both the two triggering approaches and the continuous case are very closed to each other, making to seem that ISS-based triggering approach and $\dot{V}$-triggering present a good level of performance comparable to the continuous one.

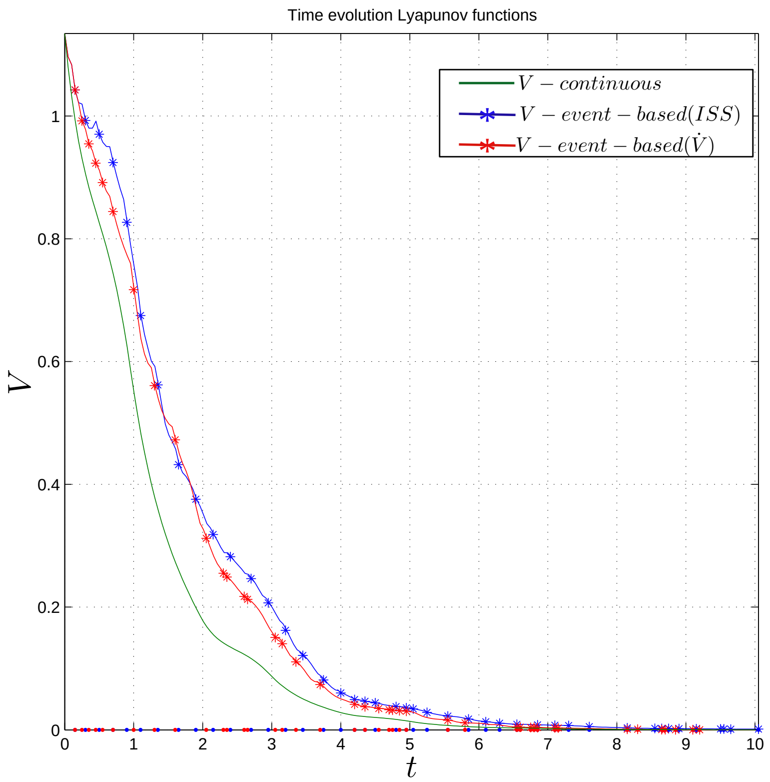


Figure 13: Lyapunov functions for both continuous and the two event-triggered cases ($\sigma = 0.9$ and $\tilde{\sigma} = 0.1$)

Final remarks

- The 3D plots in Fig. 6, 8 and 10, concerning the components of solutions, seem to be the same; but actually they are not. Such plots indeed present minor differences (for instance, a bit more of oscillations) with respect to each other even with respect to components of solution in continuous case. Therefore, it means that solutions of linear hyperbolic systems in even-based control do not deviates too much from a desired threshold implicitly given by the triggering condition. In addition, attractive might be evident from the numerical point of view.

  However, we were not able to prove, in details, the formal existence of solutions between two execution times. We strongly believe that there is no problem and in fact solution might exist everywhere except in a finite number of points of discontinuity. But it is still an open question. One way to handle this matter in future works is to turn to what is developed for switched linear hyperbolic systems. It would be inspired again in [24].

- We have mentioned in Section 2 that one of the most important issues in event-based control is the existence of a minimal inter-execution time. Therefore, in our study, the question arose naturally. Nevertheless, the difficulty to answer it lies on the discontinuity of solution. We again believe that there is no infinite number of execution time in finite time. A technical proof is not done in this instance and for future works it could be considered. But, numerically, we have observed that while reducing the time step discretization, the number of event does not increase. This is a good signal that *zeno-phenomena* is avoided.

- One way to reduce the magnitude of peaks aforementioned, is to reduce the time-step when discretizing the time in the numerical scheme. Although it is possible to reduce the peak value, one will never get an equality in the triggering conditions. At each time a discontinuity appears, an event may be generated. So, such a discontinuity enforces the control execution task. Also events may happen consecutively.

- Due to the discontinuity of solution we have worked in L^2-norm. In that framework, one does not have the compatibility conditions.

- Although the rate of convergence for both triggering strategies is slower than in continuous case, a suitable level of performance holds. It means that while reducing the number of execution times, we can achieve a system behaviour quite similar to the ideal case when continuous controller is applied periodically.

- Initial conditions influence a lot the triggering algorithm, specially if the zero-order compatibility condition is satisfied.

4 Conclusions and perspectives

The combination of an emerging field such as event-based control with a very interesting but at the same time difficult field such as control theory of PDEs, promises to exploit interesting topics in applied mathematics and automatic control. In this document we have stated the first stages of contribution so that a potential theory of event-based control for systems modelled by infinite-dimensional rises with important applications to the industry. In consequence, we have started designing a event-based boundary control for 1-dimensional linear hyperbolic systems. Our analysis relies on the main ideas of event-based control already carried out for linear and nonlinear systems. We were able to define two triggering conditions guaranteeing stability and a suitable level of performance. Indeed, we have presented the essential of event-based control for finite-dimensional systems with the aim to extend them to infinite case. We have then taken advantage of sufficient conditions on the boundary for stabilization of linear hyperbolic system thanks to Lyapunov techniques concluding that the combination between event-based and boundary control was possible and carried out.

Our preliminary results suggest that our approach can be powerful when reducing computational costs in a context where sensors and actuators may be distributed in a network modelled by infinite-dimensional systems and controlled on some of the boundaries.

During the project, several difficulties appeared, for example when implementing the trigger algorithm. Some of them were overcome. Also, several questions arose and were not answered. They are still open questions which also motive us to study this field. Accordingly, future investigation lines may be centered in:

- improving the numerical scheme to compute exactly the reset time, in the event-triggered algorithm.
- studying the existence and uniqueness of solutions for hyperbolic system between two execution times.
- studying the strong influence of initial conditions in the trigger algorithm.
- studying the existence of the minimal inter-sampling time.
- defining an event-triggering condition from backstepping stabilization approach for hyperbolic systems (e.g [18] and [17]).
- potential applications such as flow control.
- extension to other partial differential equation such as parabolic systems.

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APPENDICES

A More about Hyperbolic Systems

A.1 Linear hyperbolic systems of conservation laws

The sequel is based on [8].

Definition 3. *Let us consider $N \geq 1$ an integer, $A \in \mathbb{R}^{N \times N}$ a real matrix. The system of conservation laws*

$$\partial_t y + A \partial_x y = 0 \quad x \in \mathbb{R}, \quad t \leq 0 \tag{A.1}$$

being $[0\infty) \times \mathbb{R} \ni (x,t) \mapsto y(t,x)$, is said to be hyperbolic if the matrix A is diagonalizable on $\mathbb{R}$; i.e. there exist real eigenvalues

$$\lambda_1 \leq \lambda_1 \leq ... \leq \lambda_k \leq \lambda_{k+1} \leq ... \leq \lambda_N$$

and eigenvectors $r_k \in \mathbb{R}^N$ such that

$$A r_k = \lambda_k r_k$$

forming a base for $\mathbb{R}^N$.

In the case of a 1-dimensional linear hyperbolic system, one can explicitly set up the solution by decomposing the unknown vector $y(\cdot,\cdot)$ on the base of vectors r_k as follows:

$$y = \sum_{k=1}^{N} \varphi_k(t,x) r_k \tag{A.2}$$

$\varphi(\cdot,\cdot)$ are also called the *characteristic variables.* They are solution of N advection equations with speed λ_k:

$$\partial_t \varphi_k + \lambda_k \partial_x \varphi_k = 0 \tag{A.3}$$

Indeed, by replacing Equation (A.2) into (A.1), we get

$$\partial_t y + A \partial_x y = \sum_k \left(\partial_t \varphi_k r_k + \partial_x \varphi_k A r_k \right) = \sum_k \left(\partial_t \varphi_k + \lambda_k \partial_x \varphi_k \right) r_k$$

In order to compute the solution $y(\cdot,\cdot)$ at point (t,x), we decompose the unknown vector on the base $(r_k)_k$ of eigenvectors of A. Then, the k^0 characteristic variable ϕ_k is solution of one advection Equation (A.3) which has constants solutions along the characteristic curves of equation $\frac{dX}{dt} = \lambda_k$. Therefore,

$$\phi_k(t,x) = \phi_k^0(x - \lambda_k t), \quad 1 \leq k \leq N \tag{A.4}$$

where the initial condition $y^0(\gamma)$ is supposed to be decomposed according to (A.2):

$$y^0(\gamma) = \sum_{k=1}^{N} \varphi_k^0(\gamma) r_k, \quad \gamma \in \mathbb{R} \tag{A.5}$$

Finally, by regrouping (A.4) and (A.2) we have,

$$y(t,x) = \sum_{k=1}^{N} \varphi_k^0(x - \lambda_k t) r_k \tag{A.6}$$

A.2 Theoretical remark

Let us just remark that both ISS-based triggering condition and $\dot{V}$-based triggering condition rely on the Lyapunov function given by

$$V(y) = \int_0^1 y(x)^T \mathcal{Q}(x) y(x) dx$$

which, in turn, is computed thanks to solution $y(t,x)$. It means that the knowledge of solution along the one-dimensional space $[0,1]$ is required. However, in practice, to be able to compute it online, one would need to put sensors everywhere along $[0,1]$ which is indeed inappropriate or even useless. This is a motivation for which we seek for obtaining information about $y(t,x)$ but just from the boundaries.

In order to illustrate the ideas, let us first consider the simplest case, i.e the a linear hyperbolic equation of conservation laws given in Riemann coordinates.

$$\partial_t y + \lambda \partial_x y = 0 \tag{A.7}$$

so, $y \in \mathbb{R}$. It is known from the characteristics method that the solution is given by:

$$y(t,x) = g(x - \lambda t) \qquad \textit{traveling wave}$$

for any smooth function g of one variable. The verification that it is certainly a solution is straightforward. Moreover, it can be also shown that

$$y(t,x) = y\left(t - \frac{x}{\lambda}, 0\right) = g\left(t - \frac{x}{\lambda}\right) \tag{A.8}$$

is also a solution of (A.7). We use the fact that from the information $y(t - \frac{x}{\lambda})$, one is able to recover the information about $y(t,x)$. This is virtue of $y(t,x)$ is invariant along the characteristic curve starting in $y(t - \frac{x}{\lambda}, 0)$. Hence, it is possible to re-compute the Lyapunov function as follows:

$$V(y) = \int_0^1 g\left(t - \frac{x}{\lambda}\right)^T \mathcal{Q}(x) g\left(t - \frac{x}{\lambda}\right) dx \tag{A.9}$$

Let us now consider a 2×2 linear hyperbolic system of conservation laws.

$$\partial_t y + \Lambda \partial_x y = 0 \tag{A.10}$$

- Case 1: $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. In this case, the two waves travel at the same speed. So, it is enough to compute $V(y)$ from (A.9).

- Case 2: $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. In this case, since it is possible to decouple the two equations, that is

$$\begin{aligned} \partial_t y_1 + \lambda_1 \partial_2 y_1 &= 0 \\ \partial_t y_2 + \lambda_2 \partial_2 y_2 &= 0 \end{aligned}$$

we can use the information $y(t - \frac{x}{\lambda_1}, 0)$ for the first equation and $y(t - \frac{x}{\lambda_2}, 1)$ for the second one. Hence, the Lyapunov function would be something of the form

$$V(y) = \int_0^1 \begin{bmatrix} y_1(t - \frac{x}{\lambda_1}, 0) \\ y_2(t - \frac{x}{\lambda_2}, 1) \end{bmatrix}^T \mathcal{Q}(x) \begin{bmatrix} y_1(t - \frac{x}{\lambda_1}, 0) \\ y_2(t - \frac{x}{\lambda_2}, 1) \end{bmatrix} dx$$

On the other hand, if we deal with a linear hyperbolic system of balance laws, we can proceed similarly .

$$\partial_t y + \lambda \partial_x y = Fy$$

To illustrate the idea, let us only consider $y \in \mathbb{R}$. Now, we seek for a solution $y(t,x) = y(t + \frac{x}{\lambda}, 0) = g(t + \frac{x}{\lambda})$. In that way, it is clear that

$$2g'\left(t + \frac{x}{\lambda}\right) = Fg\left(t + \frac{x}{\lambda}\right)$$

Therefore, the solution is given by:

$$g(t + \frac{x}{\lambda}) = e^{\frac{1}{2}F\left(t + \frac{x}{\lambda}\right)} g(0) \tag{A.11}$$

and hence, the Lyapunov function is then re-computed as follows

$$V(y) = \int_0^1 \left(g(0) e^{\frac{1}{2}F\left(t+\frac{x}{\lambda}\right)}\right)^T \mathcal{Q}(x) \left(g(0) e^{\frac{1}{2}F\left(t+\frac{x}{\lambda}\right)}\right) dx$$

A.3 LxF numerical scheme

The Lax-Friedrichs (LxF) method is used for the numerical solution of hyperbolic partial differential equations. This scheme is obtained from the explicit centered method. To illustrate this, let us just consider the simplest case, i.e. the advection equation for which a explicit centred method is applied.

$$\frac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) + \frac{a}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \tag{A.12}$$

In practice this method is not useful due to stability considerations. However, a minor modifications gives a more useful method. By replacing u_j^n in the time derivative term by $\frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)$, we obtain the *Lax-Friedrichs method*,

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) \tag{A.13}$$

Regarding the stability issues, it can be verified by means of Von Neumann - Fourier method that the condition of the scheme to be stable is given by [8]:

$$\left|\frac{a\Delta t}{\Delta x}\right| \leq 1$$

A.4 About the size of F for a linear hyperbolic system of balance laws

It is very interesting to point out that F being sufficiently small together with dissipative condition for stability (3.5), certainly get exponential stability. In view of [5], the system of balance law is regarded as a system *almost* conservative. There, theorem 1 states that if $\rho_1(G) < 1$, there exists $\epsilon > 0$ such that, if $\|F\| < \epsilon$, then the linear hyperbolic system is exponentially stable. In the proof (see again [5]), based on Lyapunov techniques as usual, $\|F\| < \epsilon$ is imposed so as to get a strict Lyapunov function an hence, the exponentially convergence of solutions of the system to zero in L^2-norm. Furthermore, in [26], the source term is considered as a perturbation. A sufficient criterion is then given in terms of the boundary condition to get robust stabilization; but the characteristic method is used instead of Lyapunov techniques. The existence of an upper bound for the norm of the source term is also shown.

It is worth remarking that whether F is not small enough, one cannot conclude anything about the exponential stability. This is why, F small is considered as a sufficient condition as well. Let us show an example (borrowed and modified from [23] in Appendix A) illustrating the above fact. So, let us consider the following specific 2×2 hyperbolic system of conservation laws:

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0, \quad x \in [0,1], t \geq 0 \tag{A.14}$$

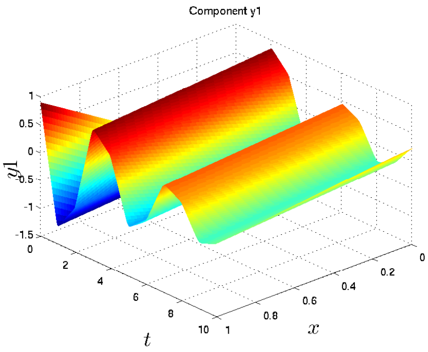
where $\Lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 0 & -1.2 \\ 0.6 & 0 \end{pmatrix}$ and initial conditions $y(0,x) = \begin{pmatrix} 2.2x - 1.2 \\ 0.4x + 0.6 \end{pmatrix}$ satisfying the zero-order compatibility condition.

Fig. A.14 shows the result of simulations where indeed the convergence is clear.

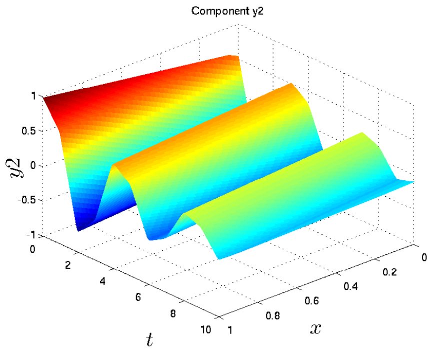
Now, if we consider the particular case of a system of balance laws when $\Lambda = 0$, i.e

$$\partial_t y(t,x) = F y(t,x)$$

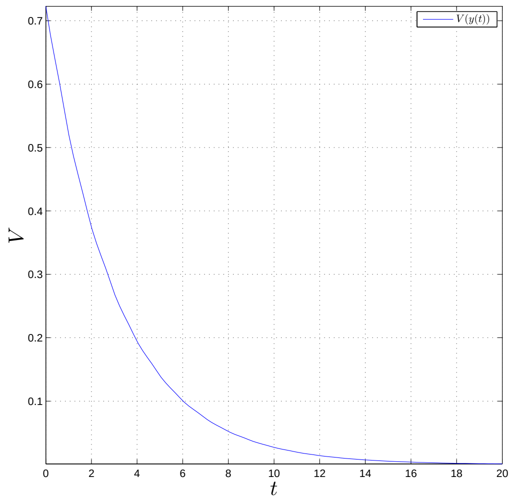
It is a ordinary differential equation where x is viewed as a parameter. $F = \begin{pmatrix} -4 & 5 \\ -3 & 3 \end{pmatrix}$ It can be easily checked that F has their eigenvalues in negative real part. Therefore, since it is Hurwitz, the system is exponentially stable. See Fig. A.15.



(a) Component y_1

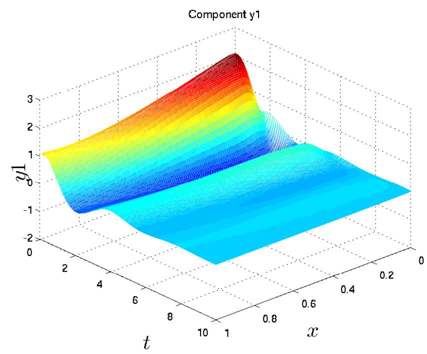


(b) Component y_2

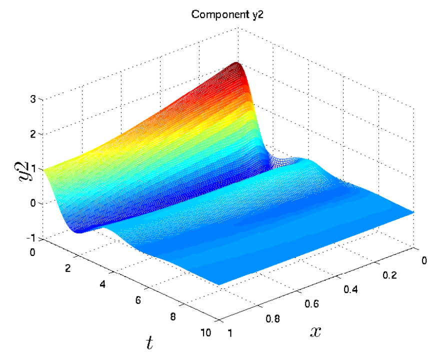


(c) Lyapunov functions

Figure A.14: Time evolution of the first component y_1 (a) and of the second component y_2 for the system of conservation laws. (c) Lyapunov functions comparison.



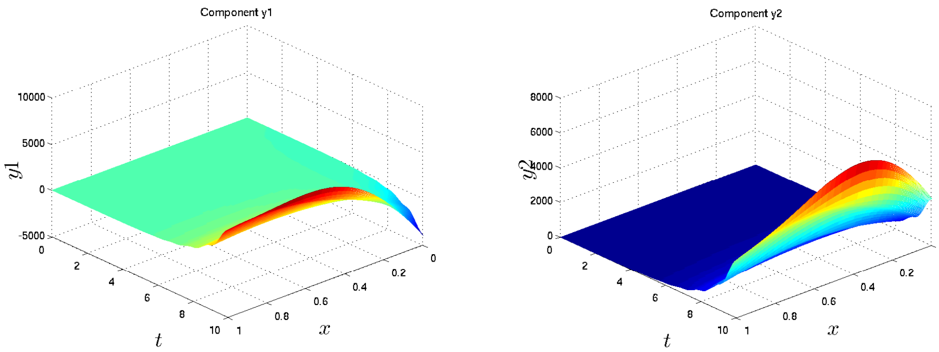
(a) Component y_1



(b) Component y_2

Figure A.15: Time evolution of the first component y_1 (a) and of the second component y_2 for the finite dimensional system.

Now, combining the two previuos systems leads to a system of balance laws $\partial_t y(t,x) + \Lambda\partial_x y(t,x) = Fy(t,x)$. This system seems to be unstable as is shown in Fig. A.16



(a) Component y_1 (b) Component y_2

Figure A.16: Time evolution of the first component y_1 (a) and of the second component y_2 for the system of balance laws.

Estimation of ϵ

Now, we shall study a simple case to estimate the value of ϵ. Let us consider

$$\partial_t y(t,x) + \Lambda\partial_x y(t,x) = Fy(t,x) \tag{A.15}$$

$$y(t,0) = Gy(t,1) \tag{A.16}$$

where $Fy(t,x)$ is viewed as a perturbation. $V(y) = \int_0^1 y(x)^T e^{-2\mu x} Q y(x)dx$. The time derivative when particular case $m = 0$ and $n = 2$ yields

$$\dot{V} = \int_0^1 -[\partial_x(y^T e^{-2\mu x} Q\Lambda y) + 2\mu y^T e^{-2\mu} Q\Lambda y]dx + \int_0^1 y^T(F^T e^{-2\mu} Q + e^{-2\mu} QF)ydx$$

$$\begin{aligned}\dot{V}(y) = &\int_0^1 y^T(-2\mu\Lambda e^{-2\mu x} Q\Lambda)ydx + y^T(t,1)\left[G^T Q\Lambda G - e^{-2\mu} Q\Lambda\right] y(t,1)\\ &+ \int_0^1 y^T(F^T e^{-2\mu x} Q + e^{-2\mu x} QF)ydx\end{aligned}$$

Therefore, if we seek for a ISS Lyapunov function, we impose the following conditions:

$$-2\mu\Lambda e^{-2\mu x} Q\Lambda \leq -2\nu e^{-2\mu x} Q$$
$$G^T Q\Lambda G \leq e^{-2\mu} Q\Lambda$$

and we take into account the influence of perturbation term, $Fy(t,x) = \delta(t,x)$ (distributed perturbation: closely related to what is presented in Appendix B.2). So, we finally get a ISS-Lyapunov function as follows:

$$\begin{aligned}\dot{V}(y) &\leq -2\nu V + \int_0^1 y^T e^{-2\mu x} Q\delta + \delta^T e^{-2\mu x} Q y dx \\ &= -2\nu V + 2\int_0^1 y^T e^{-2\mu x} Q\delta dx\end{aligned}$$

$$\Rightarrow \dot{V}(y) \leq -2\nu V + \int_0^1 e^{-2\mu x}\left(\frac{1}{\alpha} y^T Q y + \alpha\delta^T Q\delta\right) dx \tag{A.17}$$

$$\leq (-2\nu + \frac{1}{\alpha})V + \int_0^1 \lambda \|\delta\|^2 e^{-2\mu x} dx \tag{A.18}$$

for some $\alpha > 0$ and λ is the largest eigenvalue of matrix αQ.

Moreover, we look for an estimation of F size. For that purpose, we can use inequality (A.17) which yields,

$$\begin{aligned}\dot{V} &\leq -2\nu \int_0^1 e^{-2\mu x}\left(y^T Q y - \left(\frac{1}{\alpha} y^T Q y\right)\frac{1}{2\nu}\right) dx + \int_0^1 e^{-2\mu x}\alpha\left(\delta^T Q\delta\right) dx \\ &= -\alpha \int_0^1 \left[\frac{-2\nu}{\alpha}\left(e^{-2\mu x}\left(1 - \frac{1}{2\nu}\right) y^T Q y\right) - e^{-2\mu x}\delta^T Q\delta\right]\end{aligned}$$

Then, the term inside the integral must be strictly positive. Setting $e^{-2\mu x}$ bounded and knowing that $\delta = Fy$ we finally deduce the following:

$$\begin{aligned}(Fy)^T Q(Fy) &< \frac{2\nu}{\alpha}\left(1 - \frac{1}{2\nu\alpha}\right) y^T Q y \\ \Leftrightarrow y^T F^T Q F y &< \left(\frac{2\nu}{\alpha} - \frac{1}{\alpha^2}\right) y^T Q y\end{aligned}$$

Then, $F^T Q F < \left(\frac{2\nu}{\alpha} - \frac{1}{\alpha^2}\right) Q$, and hence an estimative of the size of F is done:

$$\|F\| < \sqrt{\left(\frac{2\nu}{\alpha} - \frac{1}{\alpha^2}\right)}$$

Moreover, it can be easily verified by means of a simple optimization procedure that

$$\|F\| < \nu$$

A.5 Event-trigger algorithm

Input: $y(0,x)$ (Initial conditions), $y(t,1) = (H + BK)y(t,0) + d$ (Dissipative boundary conditions with disturbance) , Thorizon, mesh discretization, other parameters depending of the triggering condition.
Output: $y(t,x)$ (solution), $V(y)$ (Lyapunov function), ev (event function)
begin
    Setting the system to be solved;
    sol = setup(form,@pdefun,t,x,Y,method,[],@bcfun) ;
    j=1;
    while $t \leq$ *Thorizon* **do**
        if *insideC = 1* **then**
            for *i = j to NT(length time mesh)* **do**
                if *insideC = 1* **then**
                    Solving pde before event-detection;
                    sol = hpde(sol,howfar,timestep);
                    t = sol.t;
                    y = sol.u;
                    Computing the measurement error;
                    $e(t) = y(t_i, 1) - y(t, 1)$;
                    $d(t)$=$BK * e(t)$;
                    computing Lyapunov function;
                    V;
                    monitoring event function ;
                    ev;
                    insideD;
                else
                    j=i;
                    insideD=1;
                    stop the integration;
                    break;
                end
            end
        end
        Updating;
        $e(t_i) = y(t_i, 1) - y(t_i, 1) = 0$;
        $d(t_i) = 0$;
        $V(y)$;
        monitoring event function ;
        ev;
        insideD;
    end
    return $y(t,x)$ $V(y)$ ev
end

Algorithm 1: Event-Trigger algorithm for linear hyperbolic systems

B Input-to-State Stability

B.1 ISS for finite-dimensional systems

The input-to-state stability (ISS) property provides a natural framework in which to formulate notions of stability with respect to input perturbations [31] [32].

The concept of ISS implies that, given a bounded input of the dynamical system, the internal states remain bounded. In other words, the main concept relies on boundedness of the responses to any bounded disturbance and the convergence to zero if disturbances vanish. A comprehensive survey on ISS concepts for finite-dimensional systems can be found in [32]. Here we point out some important aspects.

Let us consider a control system of the form:

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{B.1}$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschitz continuous on compacts if for every compact set $S \subset \mathbb{R}^n$ there exists a constant L such that

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for every x, $y \in S$. A continuous function $\alpha : [0, a) \to \mathbb{R}_{0+}$ $a > 0$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A class $\mathcal{KL}$ function is a function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each t and decrease to zero on the second argument as $t \to \infty$ [33], [31].

Definition 4. *[33] A system is said to be input-to-state stability (ISS) if there exist a function $\beta \in \mathcal{KL}$ and a function $\gamma \in \mathcal{K}_\infty$ such that*

$$\|x\| \leq \beta(\|x^0\|, t) + \gamma(\|u\|_\infty) \tag{B.2}$$

holds for all solutions.

Another formulation of ISS, by knowing that $\max\{a, b\} \leq a + b \leq \max\{2a, 2b\}$ is the following:

$$\|x\| \leq \max\{\beta(\|x^0\|, t), \gamma(\|u\|_\infty)\} \tag{B.3}$$

ISS combines overshoot and asymptotic behaviour. We recall that in the linear case: $\dot{x} = Ax + Bu$, we have that $\|x(t)\| \leq \beta(t)\|x^0\| + \gamma\|u\|_\infty$ which is the estimate according to the solution $x(t) = e^{At}x^0 + \int_0^\infty e^{A(t-\tau}Bu(\tau)d\tau$. Therefore $\beta(t) = \|e^{At}\| \to 0$ and $\gamma = \|B\| \int_0^\infty \|e^{As}ds\| < \infty$. One of the characterization of ISS which also results interesting, is a dissipation notion in terms of Lyapunov-like function .

Definition 5. *[32] A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function if it is positive definite, that is $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ and proper (radially unbounded), that is $V(x) \to \infty$ as $\|x\| \to \infty$. Accordingly $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function if and only if there* *exists $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ such that*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$$

In addition, we recall the time derivative along the solutions of (2.1) *is given by*

$$\dot{V}(x,u) := \nabla V(x) \cdot f(x,u)$$

The fact given by (5) means, on one hand that the lower bound amounts to properness and $V(x) > 0$. On the other hand the upper bound guaratees $V(0) = 0$. See [16] where the above characterization is presented as a lemma with its respective proof. As a particular case, for a quadratic positive definite function $V(x) = x^T P x$, we know that $\lambda_{\min}(P)\|x\|^2 \leq V(x) = x^T P x \leq \lambda_{\max}(P)\|x\|^2$.

Definition 6. *[32] ISS-Lyapunov function*
An ISS-Lyapunov function for (B.1) *is by definition a smooth storage function V for which there exists functions γ, $\alpha \in \mathcal{K}_\infty$ so that*

$$\dot{V}(x,u) \leq -\alpha(\|x\|) + \gamma(\|u\|) \qquad \forall x, u \tag{B.4}$$

Integrating (B.4), we have the following *dissipation inequality*

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} w(u(s), x(s))ds \tag{B.5}$$

The supply function is $w(u,x) \leq \gamma(\|u\|) - \alpha(\|x\|)$.

Finally three more remarks: According to Massera's theorem, *Global asymptotic stability GAS* is equivalent to the existence of smooth Lyapunov function. A system is ISS if and only if it admits a smooth ISS lyapunov function (which is proper and positive definite, solution of the differential inequality (B.4)). For nonlinear finite-dimensional systems, ISS implies asymptotic stability. But the converse is false.

B.2 ISS for Infinite-dimensional systems

Particularly, the aim of this section is to see the definition of a ISS-Lyapunov function for hyperbolic systems. The sequel is based on [22] and [25]. To illustrate the ideas, let us consider a linear hyperbolic partial differential equation of the form

$$\partial_t y(x,t) + \Lambda \partial_x y(x,t) = F(x,t)y(x,t) + \delta(x,t) \tag{B.6}$$

where $x \in [0, L]$, $t \in [0, \infty)$ and Λ =Diag$(\lambda_1, ..., \lambda_n)$ is a diagonal matrix in $\mathbb{R}^{n \times n}$. δ is a disturbance of class C^1.

Definition 7. *Let $v : L^2(0,L) \times \mathbb{R} \to \mathbb{R}$ be a continuosly differentiable function, periodic with respect to its second argument. The function v is said to be a weak Lyapunov function for* *the system* (B.6)*, if there are two functions k_s and k_M of class $\mathcal{K}_\infty$ such that for all functions $\phi \in L^2(0, L)$ and for all $t \in [0, \infty)$*

$$k_s(\|\phi\|_{L^2(0,L)}{}^3) \leq v(\phi, t) \leq \int_0^L k_M(|\phi(z)|)dz$$

and, when δ is identically equal to zero, for all solutions of (B.6) *and all $t \geq 0$,*

$$\frac{dv(y(\cdot, t), t)}{dt} \leq 0$$

The function v is said to be a strict Lyapunov function for (B.6) *if, in the absence of δ, there exits a real number $\lambda_1 > 0$ such that, for all solutions of satisfying* (B.6)*, and for all $t \geq 0$,*

$$\frac{dv(y(\cdot, t), t)}{dt} \leq -\lambda_1 v(y(\cdot, t), t)$$

The function v is said to be an ISS-Lyapunov function for (B.6) *if there exist $\lambda_1 > 0$ and a function λ_2 of class $\mathcal{K}$ such that, for all continuous functions δ, for all solutions of* (B.6)*, and for all $t \geq 0$,*

$$\frac{dv(y(\cdot, t), t)}{dt} \leq \lambda_1 v(y(\cdot, t), t) + \int_0^L \lambda_2(|\delta(x, t)|)dx$$

C Internship workplace: GIPSA-Lab

This internship has been carried out at Gipsa-Lab in SySco team at the control department, under the supervision of Dr. Antoine Girard, Nicolas Marchand and Christophe Prieur.

The GIPSA-lab (Grenoble Images Parole Signal Automatique) is a research institution that belongs to the French CNRS (Centre National de la Recherche Scientifique), the INPG (Institut National Polytechnique de Grenoble), the UJF (University Joseph Fourier) and the Stendhal university. It is located within the INPG site (Grenoble, France)

[3] $\|\phi\|_{L^2(0,L)} = \sqrt{\int_0^L |\phi(x)|^2 dx}$