Event-based boundary control of networks of conservation laws

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This work has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01)

1D-Hyperbolic partial differential equations

Modeling of physical networks

- **Hydraulic: Saint-Venant equations for open channels [Bastin, Coron, and d'Andréa-Novel; 2008];**
- **Road traffic [Coclite, Garavello and Piccoli; 2005];**
- **Data/communication: Packets flow on telecommunication networks [D'Apice, Manzo and Piccoli; 2006].**
- Gas pipeline : Euler equations [Gugat, Dick and Leugering; 2011];
- Electrical lines : Transmission and wave propagation [Magnusson,Weisshaar,Tripath and Alexander; 2000];
- ...

Event-based boundary control of these applications

- To propose a framework for event-based control of hyperbolic systems.
  - **A rigorous way to implement digitally continuous time controllers for hyperbolic systems.**
  - To reduce control and communication constraints.

Outline

1. Networks of conservation laws
   - Fluid-flow modeling
   - ISS stability
2. Event-based control of linear hyperbolic systems
   - Event-based stabilization
3. Conclusion and Perspectives

Fluid-flow modeling - communication networks
Compartmental representation

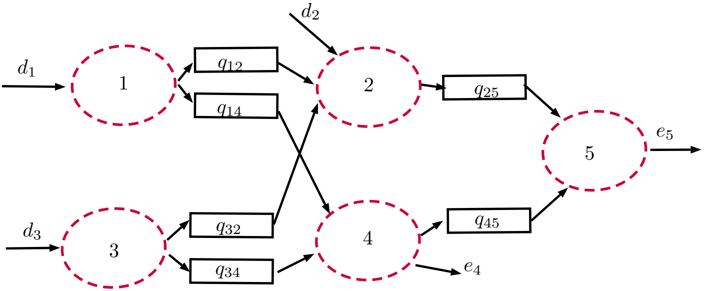

Figure: Example of a compartmental network.

1. $\mathcal{I}_n$ is the set of the number of compartments, numbered from 1 to n.
2. $\mathcal{D}_i \subset \mathcal{I}_n$ is the index set of downstream compartments connected directly to compartment i (i.e. those compartments receiving flow from compartment i).
3. $\mathcal{U}_i \subset \mathcal{I}_n$ is the index set of upstream compartments connected directly to compartment i (i.e. those compartments sending flow to compartment i).

Transmission lines

Transmission lines may be modeled by the following conservation laws [D'Apice, Manzo, Piccoli; 2008]:

$$\partial_t \rho_{ij}(t,x) + \partial_x f_{ij}(\rho_{ij}(t,x)) = 0$$

- $\rho_{ij}(t,x)$ is the density of packets;
- $f_{ij}(\rho_{ij}(t,x))$ is the flow of packets, $x \in [0,1], t \in \mathbb{R}^+, i \in \mathcal{I}_n, \quad j \in \mathcal{D}_i$.

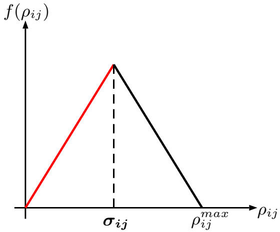

Figure: Fundamental triangular diagram of flow-density

$$f_{ij}(\rho_{ij}) = \begin{cases} \lambda_{ij}\rho_{ij}, & \text{if } 0 \le \rho_{ij} \le \boldsymbol{\sigma_{ij}} \\ \lambda_{ij}(2\sigma_{ij} - \rho_{ij}), & \text{if } \boldsymbol{\sigma_{ij}} \le \rho_{ij} \le \rho_{ij}^{max} \end{cases}$$

- $\boldsymbol{\sigma_{ij}}$ is the critical density - free flow zone and congested zone.
- λ_{ij} is the average velocity of packets traveling through the transmission line.

- We focus on the case in which the network operates in **free-flow**, i.e.

$$f_{ij}(\rho_{ij}) = \lambda_{ij} \cdot \rho_{ij}$$

for $0 \leq \rho_{ij} \leq \sigma_{ij}$.

- Let us denote the flow $f_{ij}(\rho_{ij}) := q_{ij}$.
- We rewrite the conservation laws as *Kinematic wave equation* [Bastin, Coron, d'Andréa-Novel; 2008]:

Linear hyperbolic equation of conservation laws.

$$\partial_t q_{ij}(t,x) + \lambda_{ij} \partial_x q_{ij}(t,x) = 0$$

Servers: Buffers and routers

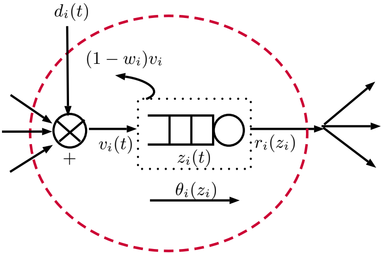

Figure: Compartment: buffer.

Dynamics for each buffer $i \in \mathcal{I}_n$(see e.g. Congestion control in compartmental network systems [Bastin, Guffens; 2006]):

$$\dot{z}_i(t) = v_i(t) - r_i(z_i(t))$$

- v_i is the sum of all input flows getting into the buffer;
- r_i is the output flow of the buffer (*processing rate function*) .

with $v_i(t) = d_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} q_{ki}(t, 1)$.

Control functions and dynamic boundary condition

Control functions

1. w_i: To modulate the input flow v_i and reject information.
2. u_{ij}: To split the flow through different lines.

$$\dot{z}_i(t) = w_i(t)d_i(t) + \sum_{\substack{k \neq i \\ k \in \mathcal{U}_i}} w_i(t)q_{ki}(t,1) - r_i(z_i(t)), \quad w_i(t) \in [0,1]$$

Dynamic boundary condition

$$q_{ij}(t,0) = u_{ij}(t)r_i(z_i(t))$$

- Splitting control (routing control): $u_{ij}(t) \in [0,1]$, $j \in \mathcal{D}_i, i \in \mathcal{I}_n$.

The output function for each output compartment $i \in \mathcal{I}_{out}$ is given by

$$e_i(t) = u_i(t)r_i(z_i(t))$$

with $\sum_{\substack{i \neq j \\ j \in \mathcal{D}_i}} u_{ij}(t) + u_i(t) = 1$.

Linearized system around an optimal free-flow equilibrium point

Coupled linear hyperbolic PDE-ODE.

$$\left.\begin{array}{l} \begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) = 0 \\ \dot{Z}(t) = AZ(t) + G_y y(t,1) + B_w W(t) + \boldsymbol{D\tilde{d}(t)} \end{cases} \\ \\ \text{with dynamic boundary condition} \\ y(t,0) = G_z Z(t) + B_u U(t) \\ \\ \text{and initial condition} \\ y(0,x) = y^0(x), \quad x \in [0,1] \\ Z(0) = Z^0. \end{array}\right\} \mathcal{P}$$

Input-to-State stability ISS

The system $\mathcal{P}$ is Input-to-State Stable (ISS) with respect to $\tilde{\boldsymbol{d}} \in \mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$, if there exist $\nu > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every $Z^0 \in \mathbb{R}^n$, $y^0 \in L^2([0,1];\mathbb{R}^m)$, the solution satisfies, for all $t \in \mathbb{R}^+$,

$$(\|Z(t)\|^2 + \|y(t,\cdot)\|^2_{L^2([0,1],\mathbb{R}^m)}) \leq C_1 e^{-2\nu t}(\|Z^0\|^2 + \|y^0\|^2_{L^2([0,1];\mathbb{R}^m)}) + \boldsymbol{C_2} \sup_{0\leq s\leq t} \|\tilde{\boldsymbol{d}}(\boldsymbol{s})\|^2 \quad (1)$$

$\boldsymbol{C_2}$ is called the **asymptotic gain (A.g)**.

Contributions on this framework:

- Modeling of communication networks under fluid-flow and compartmental representation;
- Characterization of suitable operating points for the network;

- Open-loop analysis (Lyapunov-based):
  - Sufficient condition for ISS - LMI formulation;
  - Asymptotic gain estimation;

- Closed-loop analysis (Lyapunov-based):
  - Control synthesis to improve the performance of the network;
  - LMI formulation;
  - Control constraints.
  - Minimization of the Asymptotic gain;

$$\left.\begin{aligned} W(t) &= \begin{bmatrix} \boldsymbol{K_z} & \boldsymbol{K_y} \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix} \\ U(t) &= \begin{bmatrix} \boldsymbol{L_z} & \boldsymbol{L_y} \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix} \end{aligned}\right\} \mathcal{C}$$

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Numerical simulation

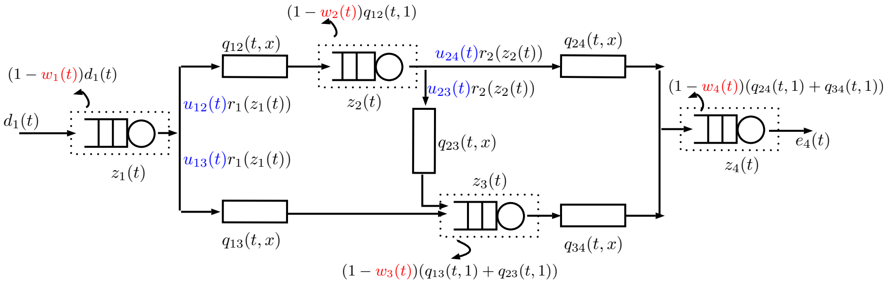
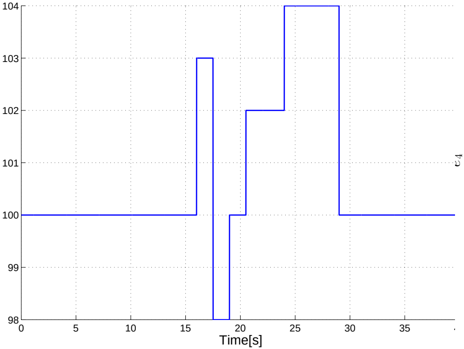
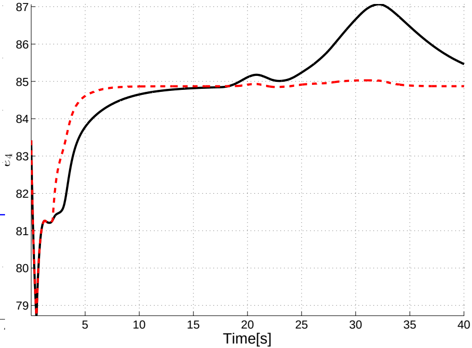


Figure: Network of compartments made up of 4 buffers and 5 transmission lines.

Exogenous input flow demand

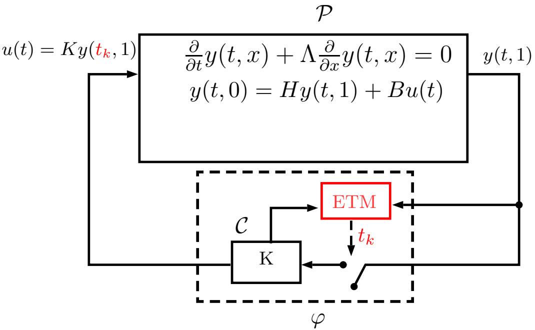


Total output flow of the network



- Asymptotic gain in open loop: 40.48
- Asymptotic gain in closed loop: 3.3

EBC of Linear hyperbolic system of conservation laws



Contributions on this framework:

- Event-triggered mechanisms (*Lyapunov-based*);

$$t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \textit{some suitable triggering condition}\}$$

  - ISS static event-based stabilization $\boldsymbol{\varphi_1}$;
  - D^+V event-based stabilization $\boldsymbol{\varphi_2}$;
  - ISS dynamic event-based stabilization $\boldsymbol{\varphi_3}$;

Conclusion and Perspectives

- Modeling and dynamic boundary control of Coupled PDE-ODE.
- Extension of event-based controls developed for finite-dimensional systems to linear hyperbolic systems by means of Lyapunov techniques;
  - **New way of sampling in time in order to implement digitally continuous time controllers for linear hyperbolic systems;**

Perspectives:

- Self-triggered implementations;
- Event-based boundary control of *parabolic* equations.

About my P.h.D

- Starting date: October 2014;
- Expected defense date: September 2017.

Publications:

Peer reviewed international journals

- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Event-based control of linear hyperbolic systems of conservation laws"*. Automatica, Vol 70, pp.275-287, 2016.

Under review international journals

- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Event-based boundary control of* 2×2 *linear hyperbolic systems via Backstepping approach"* Under review as a technical note to IEEE Transactions on Automatic Control.

Peer reviewed international conferences

- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Fluid-flow modeling and stability analysis of communication networks"*. IFAC World Congress, Toulouse, France, 2017.
- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Event-based stabilization of linear systems of conservation laws using a dynamic triggering condition"*. Proc. of the 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS), 2016.

Under review international conferences

- N. Espitia, A. Girard, N. Marchand, C. Prieur *"Dynamic boundary control synthesis of coupled PDE-ODEs for communication networks under fluid flow modeling"*. Submitted to the 56th IEEE Conference on Decision and Control (CDC) 2017.
- N. Espitia, A.Tanwani, S. Tarbouriech *"Stabilization of boundary controlled hyperbolic PDEs via Lyapunov-based event triggered sampling and quantization"*. Submitted to the 56th IEEE Conference on Decision and Control (CDC) 2017.

Thank you for your attention!

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Theorem (Control synthesis)

Let $\underline{\lambda} = \min\{\lambda_{ij}\}_{\substack{i\in\mathcal{I}_n \\ j\in\mathcal{D}_i}}$. *Assume that there exist* $\mu, \boldsymbol{\gamma} > 0$, *a symmetric positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *a diagonal positive matrix* $Q \geq I \in \mathbb{R}^{m\times m}$, *as well as control gains* $\boldsymbol{K_z}$, $\boldsymbol{K_y}$, $\boldsymbol{L_z}$ *and* $\boldsymbol{L_y}$ *of adequate dimensions, such that the following matrix inequality holds:*

$$M_c = \begin{pmatrix} M_1 & M_2 & M_3 \\ \star & M_4 & 0 \\ \star & \star & M_5 \end{pmatrix} \leq 0$$

with

- $M_1 = (A+B_w\boldsymbol{K_z})^T P + P(A+B_w\boldsymbol{K_z}) + 2\mu\underline{\lambda}P + (G_z+B_u\boldsymbol{L_z})^T Q\Lambda(G_z+B_u\boldsymbol{L_z})$;
- $M_2 = P(G_y + B_w\boldsymbol{K_y}) + (G_z + B_u\boldsymbol{L_z})^T Q\Lambda B_u\boldsymbol{L_y}$;
- $M_3 = PD$;
- $M_4 = -e^{-2\mu}Q\Lambda + \boldsymbol{L_y}^T B_u^T Q\Lambda B_u\boldsymbol{L_y}$;
- $M_5 = -\boldsymbol{\gamma} I$.

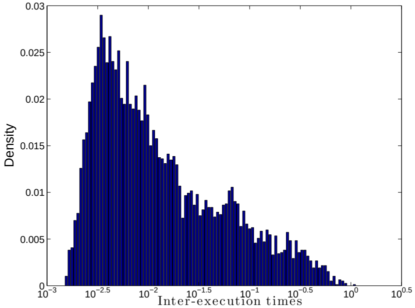
Then, the closed-loop system $\mathcal{P}$ *is ISS with respect to* $\tilde{d} \in \mathcal{C}_{pw}(\mathbb{R}^+;\mathbb{R}^n)$, *and the asymptotic gain (A.g) satisfies*

$$A.g \leq \frac{\gamma}{2\mu\underline{\lambda}} e^{2\mu}.$$

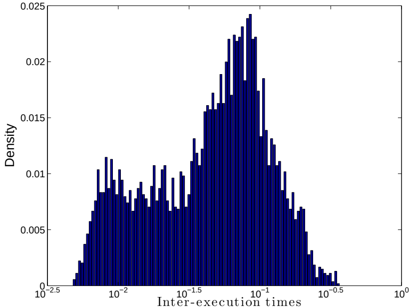
Optimization issues and control constraints

$$\begin{aligned} \textit{minimize} \quad & \frac{\gamma}{2\nu} e^{2\mu} \\ \textit{subject to} \quad & \boldsymbol{M_c} \leq 0; \\ & \|K_{zi}\| \leq \frac{p\delta_i^w}{\beta_z}; \|K_{yi}\| \leq \frac{(1-p)\delta_i^w}{\beta_y}; \|L_{zi}\| \leq \frac{\delta_{ij}^u}{\beta_z} \end{aligned}$$

- $NT = 8000$ with $\Delta t = 1 \times 10^{-3}$.



$\varphi 1$



$\varphi 3$

- Mean value of triggering times: 158.3 events;
- Mean value **inter-execution times**: 0.0432.

- Mean value of triggering times: 109.1 events;
- Mean value **inter-execution times**: 0.0640.