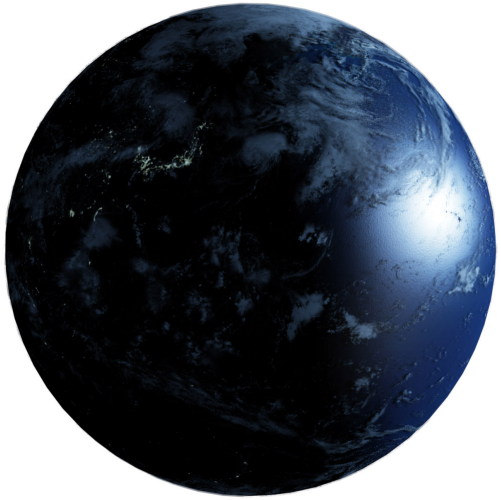
PhD Defense

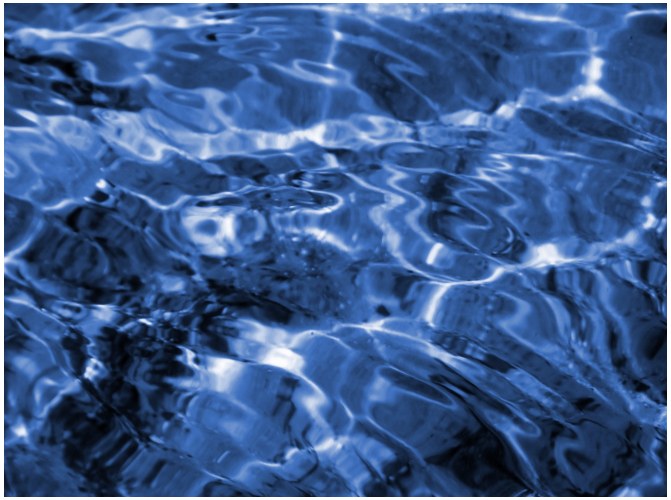
Topology and Algorithms on Combinatorial Maps

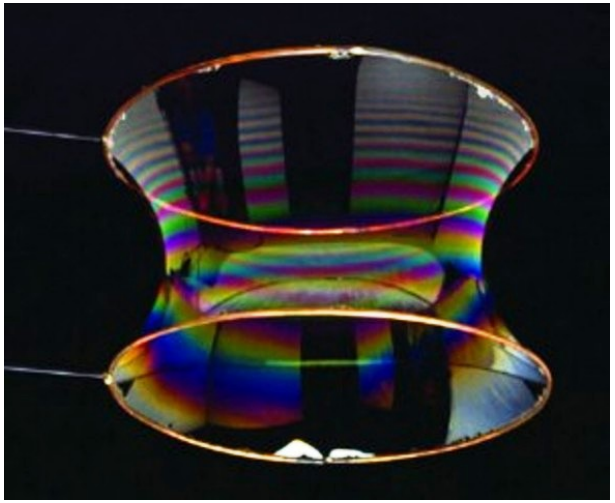
Vincent Despré

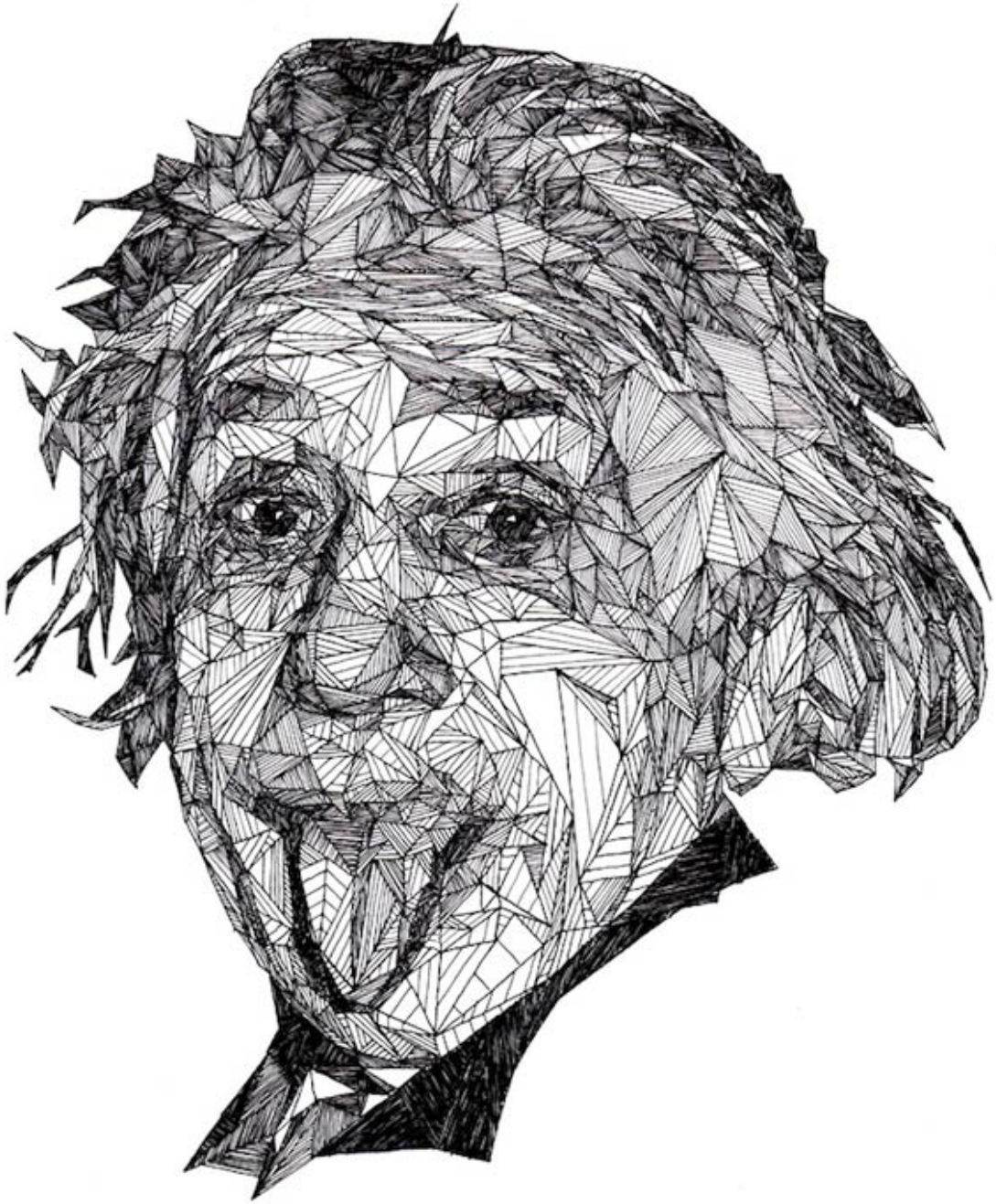
18 Octobre 2016

This work has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01) funded by the French program Investissement d'avenir.







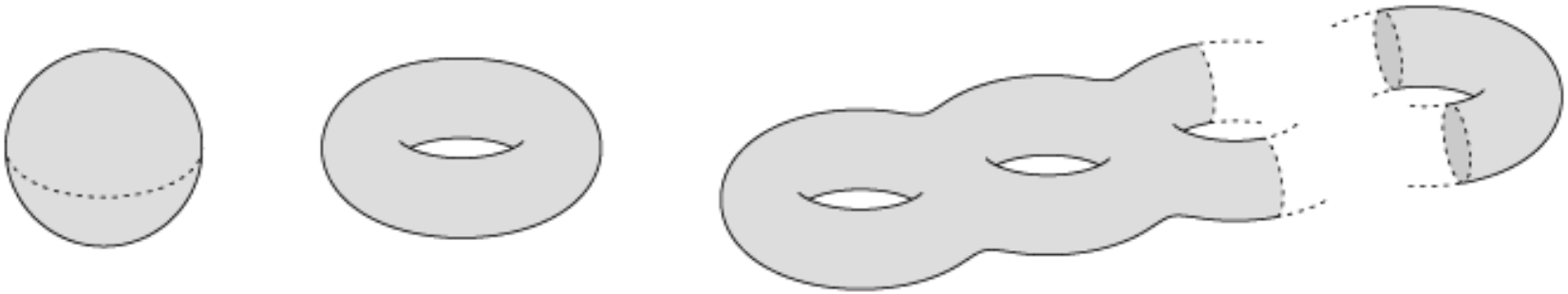


V=number of vertices, E=number of edges and F=number of faces

Euler Formula

On a surface that can be deformed to a sphere, any polygonal subdivision verifies:

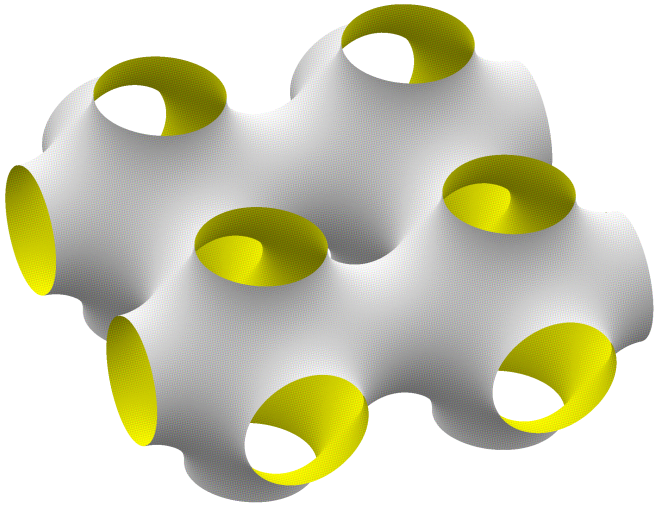
$$\chi(S) = V - E + F = 2$$



Euler Formula

On a surface S of genus g, any polygonal subdivision verifies:

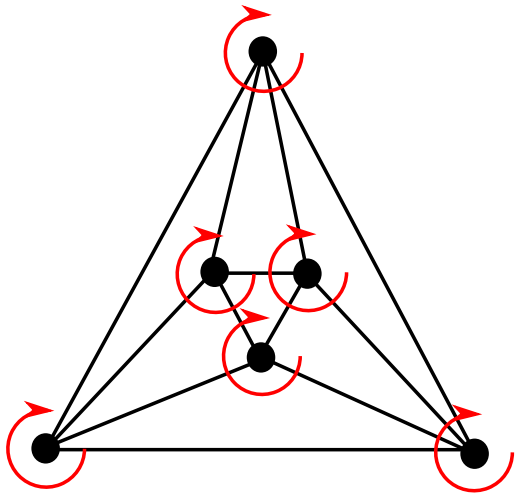
$$\chi(S) = V - E + F = 2 - 2g$$

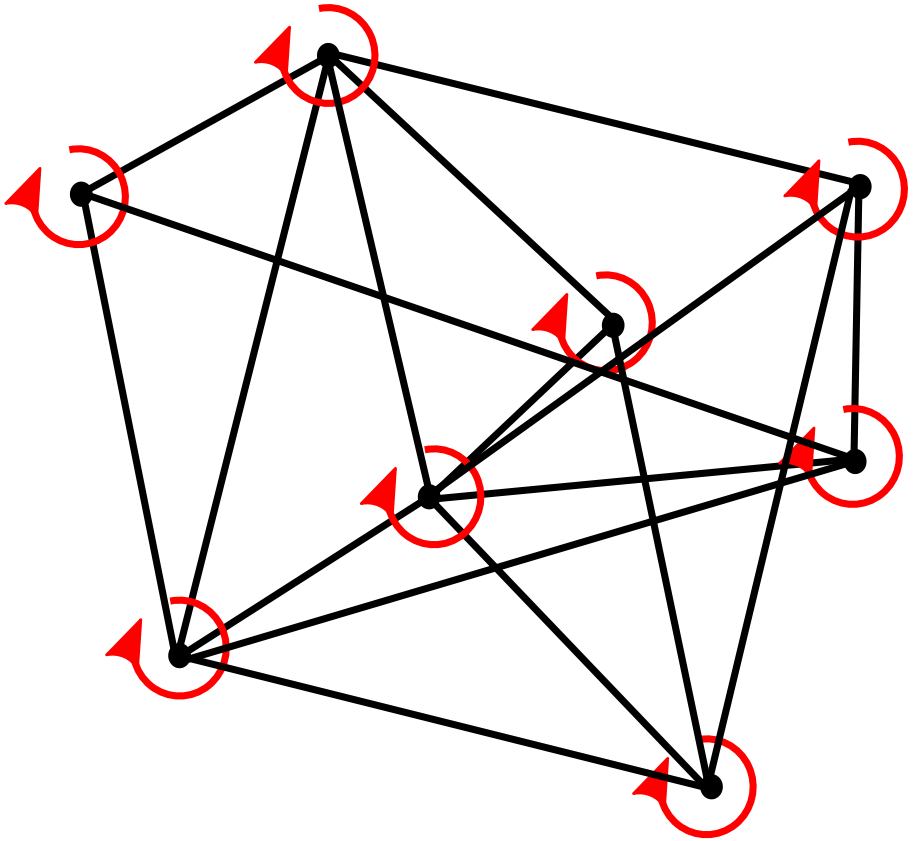


Euler Formula

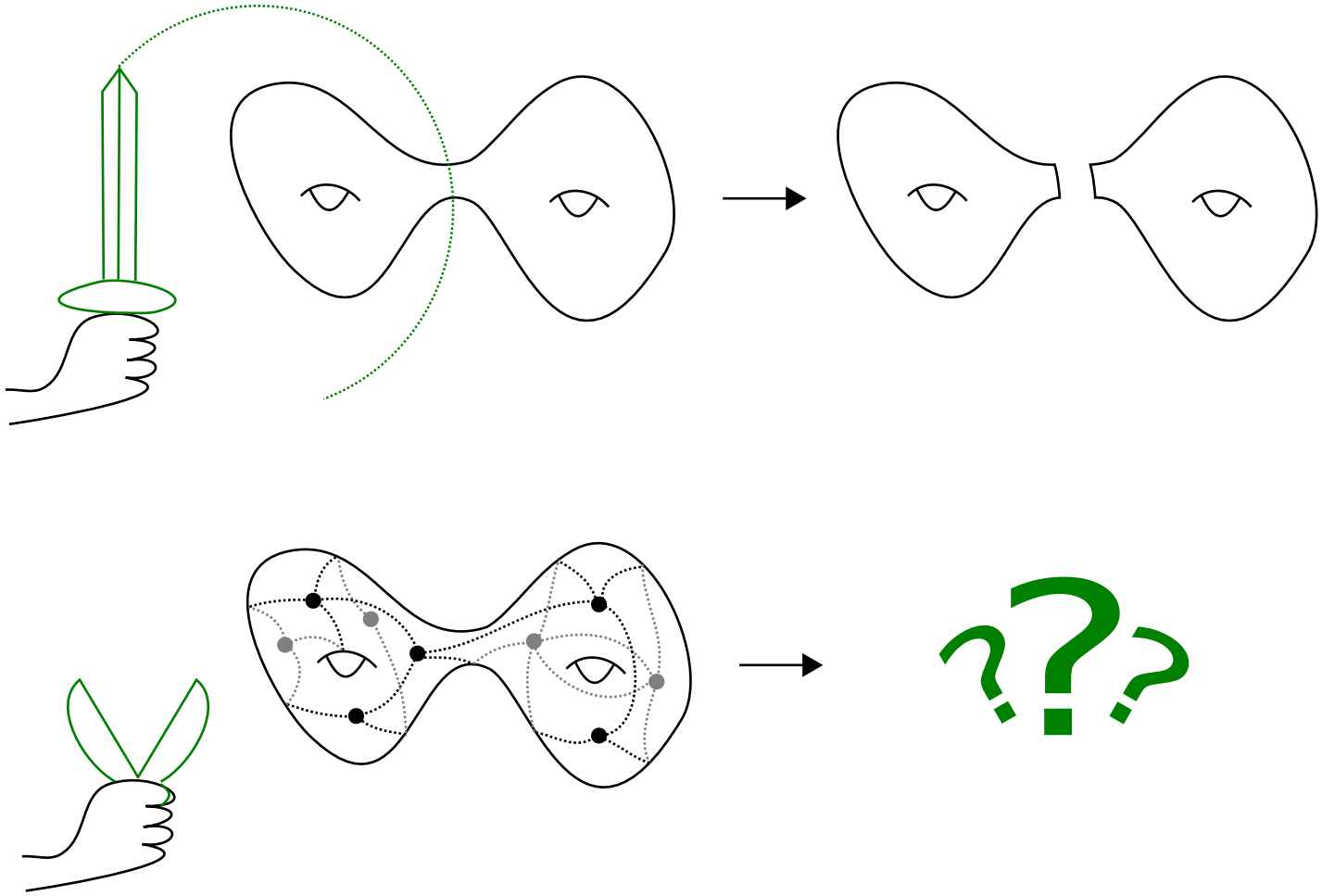
On a surface S of genus g with b boundaries, any polygonal subdivision verifies:

$$\chi(S) = V - E + F = 2 - 2g - b$$

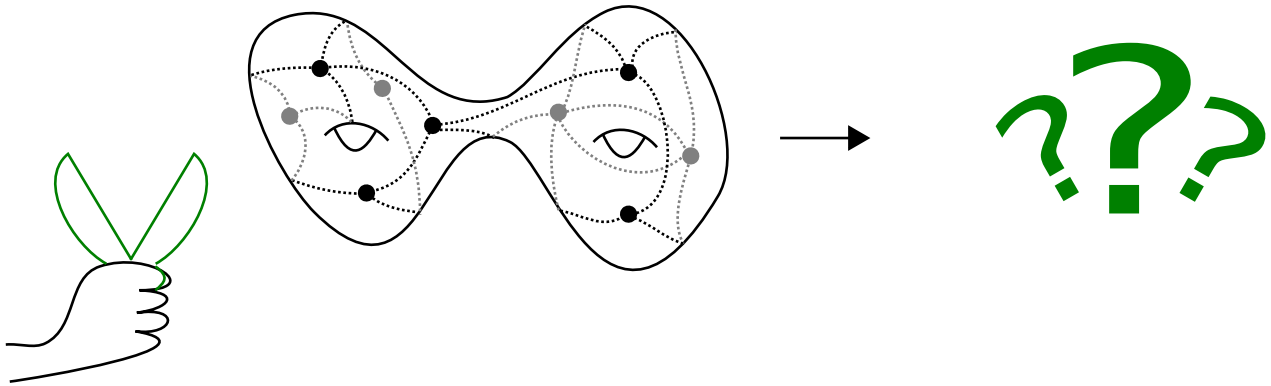




Splitting Cycles



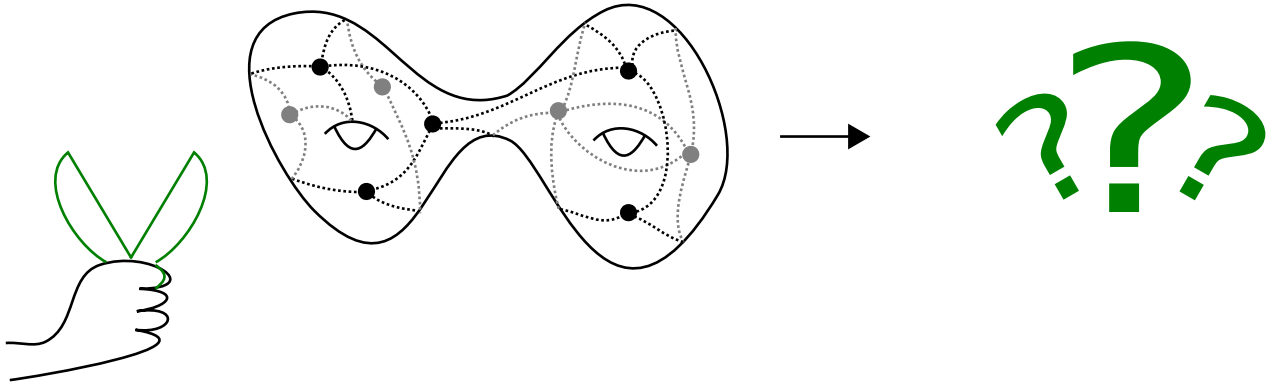
Splitting Cycles



Cabello et al. (2011)

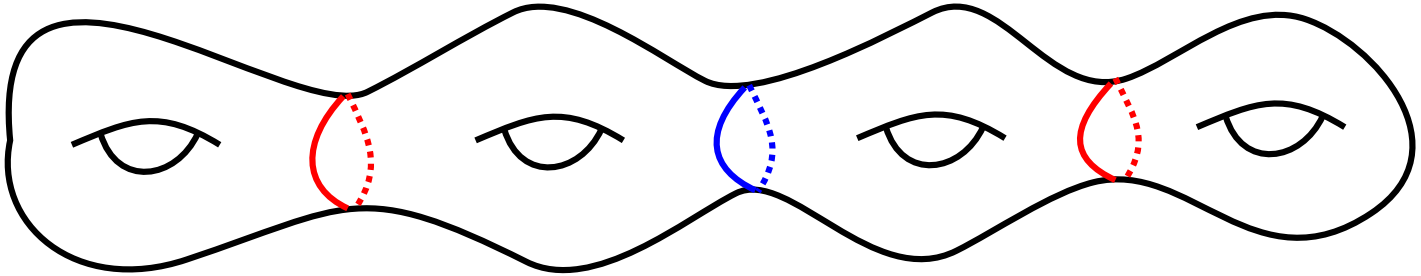
Deciding if a combiantorial map admits a splitting cycle is NP-complete.

Splitting Cycles



Barnette's Conjecture (1982)

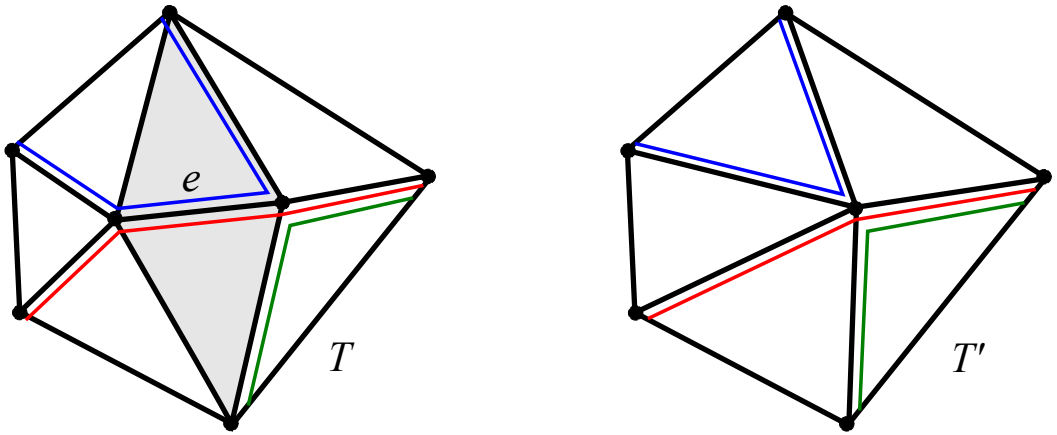
Every triangulations of surfaces of genus at least 2 admit a splitting cycle.



Conjecture (Mohar and Thomassen, 2001)

Every triangulations of surfaces of genus $g \geq 2$ admit a splitting cycle of every different type.

Irreducible Triangulations



- There are a finite number of irreducible triangulations of genus g. (Barnette and Edelson, 1988 and Joret and Wood, 2010)
- There are 396784 irreducible triangulations of genus 2.
- Unreachable for genus 3.

Genus 2 irreducible triangulations

First implementation by Thom Sulanke.

Genus 2:

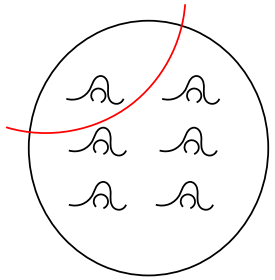
Number of triangulations: 396 784

n \ l	3	4	5	6	7	8	Average
10		2	51	681	130	1	6.09
11	2	58	2249	16138	7818	11	6.21
12	25	1516	20507	72001	22877	121	6.00
13	710	13004	50814	78059	16609	9	5.61
14	8130	30555	12308	3328	205	1	4.21
15	36794	1395	3	1	2		3.04
16	661	3					3.01
17	5						3.00

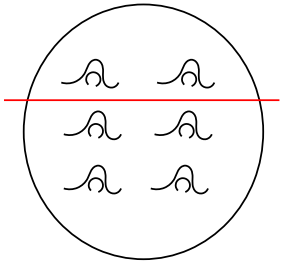
Genus 6

We consider the 59 non-isomorphic embeddings of K_{12}. (Altshuler, Bokowski and Schuchert 1996)

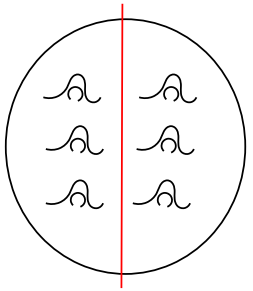
Average: 7.58
Worst-case: 8



Average: 9.41
Worst-case: 10



Average: 10.32
Worst-case: 12 (Hamiltonian cycle!)



Complete Graphs

$$\chi(S) = v - e + f = n - \frac{n(n-1)}{2} + \frac{2}{3} \cdot \frac{n(n-1)}{2} = 2 - 2g$$

$$g = \frac{(n-3)(n-4)}{12}$$

$$(n-3)(n-4) \equiv 0[12] \Leftrightarrow n \equiv 0, 3, 4 \text{ or } 7[12]$$

Theorem (Ringel and Youngs, ∼1970)

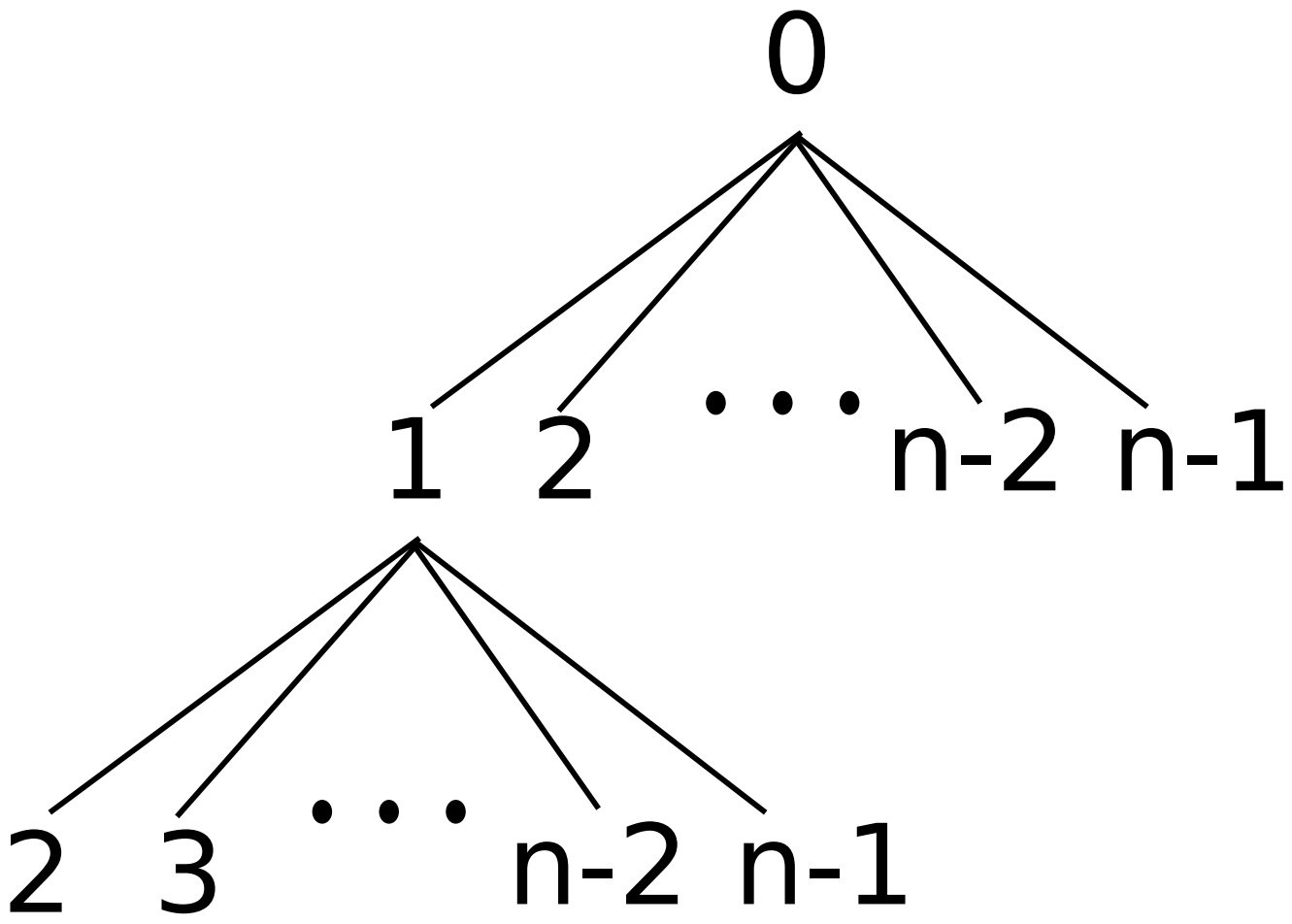
K_n can triangulate a surface if and only if $n \equiv 0, 3, 4$ or $7[12]$.

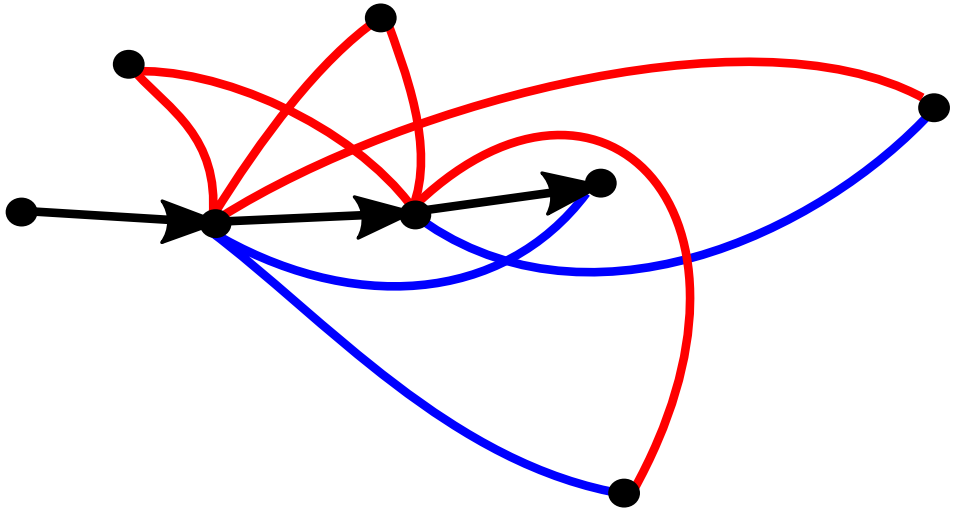
Computation time

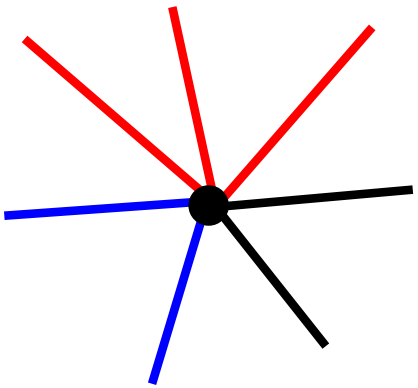
New implementation in C++. The data-structure used for the triangulations is the flag representation.

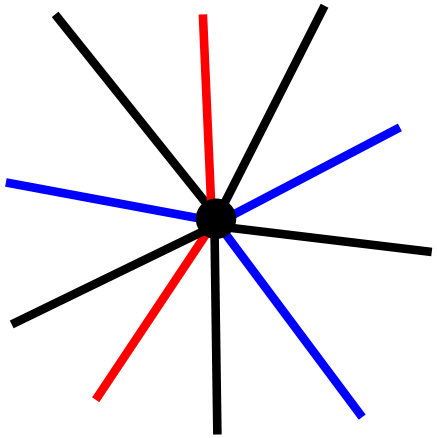
n	12	15	16	19
basic	2 s.	1 h.	12 h.	$\sim$10 years

This has been computed with an 8 cores computer with 16 Go of RAM. It uses parallel computation.


0
1
2
...
n-2
n-1
2
3
...
n-2
n-1







n	15	16	19	$\cdots$	43
basic	1 h.	12 h.	$\sim$10 years		
final	2 s.	3 s.	8 sec.		1 h.

Type \ K_n	K_{15}	K_{16}	K_{19}	K_{27}	K_{28}	K_{31}	K_{39}	K_{40}	K_{43}
1	8	10	11	12	12	8	12	10	8
2	11	12	14	16	17	13	15	15	11
3	12	14	16	19	18	15	20	18	12
4	13	16	18	20	$\perp$	17	24	19	15
5	14	16	$\perp$	27	$\perp$	20	26	24	18
6		16	$\perp$	$\perp$	$\perp$	21	30	26	20
7			$\perp$	$\perp$	$\perp$	23	32	28	21
8			$\perp$	$\perp$	$\perp$	24	$\perp$	30	23
9			$\perp$	$\perp$	$\perp$	28	$\perp$	33	24
10			$\perp$	$\perp$	$\perp$	28	$\perp$	35	25
11				$\perp$	$\perp$	29	$\perp$	36	27
12				$\perp$	$\perp$	$\perp$	$\perp$	38	29
13				$\perp$	$\perp$	$\perp$	$\perp$	40	30
14				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	31
$\vdots$				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\vdots$
29						$\perp$	$\perp$	$\perp$	42
30						$\perp$	$\perp$	$\perp$	$\perp$
max type	5	6	10	23	25	31	52	55	65

$\perp$ = No cycle found.

Counter-Examples

Mohar and Thomassen conjecture is false.

Type \ K_n	K_{15}	K_{16}	K_{19}	K_{27}	K_{28}	K_{31}	K_{39}	K_{40}	K_{43}
1	8	10	11	12	12	8	12	10	8
2	11	12	14	16	17	13	15	15	11
3	12	14	16	19	18	15	20	18	12
4	13	16	18	20	$\perp$	17	24	19	15
5	14	16	$\perp$	27	$\perp$	20	26	24	18
6		16	$\perp$	$\perp$	$\perp$	21	30	26	20
7			$\perp$	$\perp$	$\perp$	23	32	28	21
8			$\perp$	$\perp$	$\perp$	24	$\perp$	30	23
9			$\perp$	$\perp$	$\perp$	28	$\perp$	33	24
10			$\perp$	$\perp$	$\perp$	28	$\perp$	35	25
11				$\perp$	$\perp$	29	$\perp$	36	27
12				$\perp$	$\perp$	$\perp$	$\perp$	38	29
13				$\perp$	$\perp$	$\perp$	$\perp$	40	30
14				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	31
$\vdots$				$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\vdots$
29						$\perp$	$\perp$	$\perp$	42
30						$\perp$	$\perp$	$\perp$	$\perp$
max type	5	6	10	23	25	31	52	55	65

$\perp$ = No cycle found.

Conjecture

For every $\alpha > 0$, there exists a triangulation with no splitting cycles of type larger than $\alpha \cdot \frac{g}{2}$.

Encoding Toroidal Triangulations

Properties of the planar case:

1/ We have a notion of 3-orientation for triangulations.

2/ Every 3-orientation admits a unique Schnyder wood coloration.

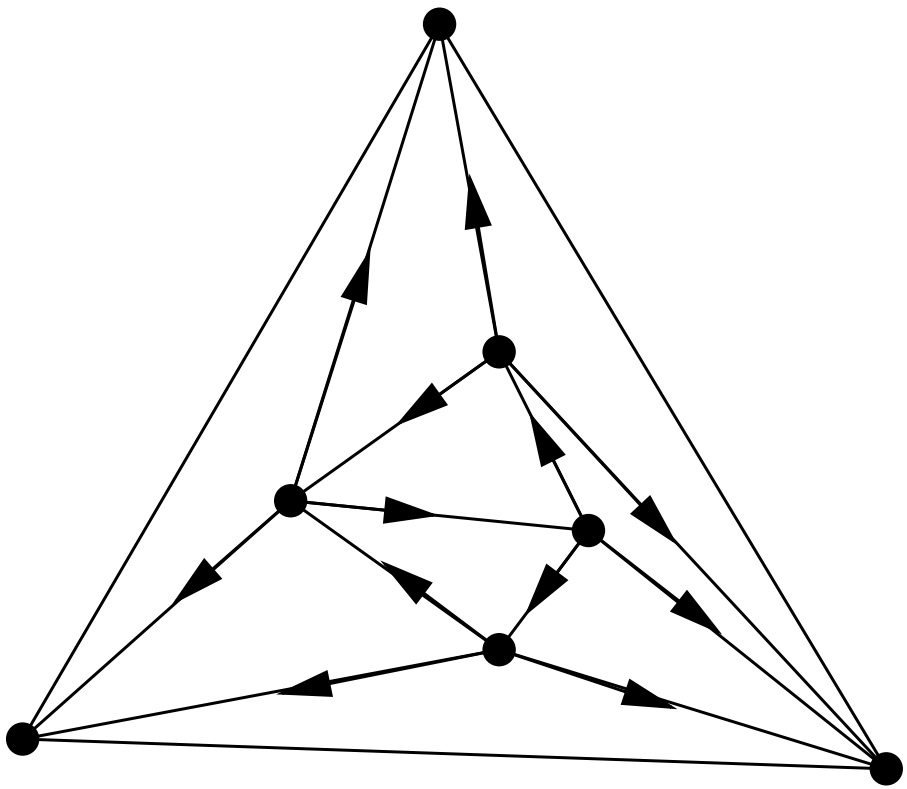
3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.

4/ The 3-orientations of a given triangulation have a structure of distributive lattice.

5/ The minimal element of the lattice has no clockwise oriented cycle.

6/ Triangulations are in bijection with a particular type of decorated embedded trees.

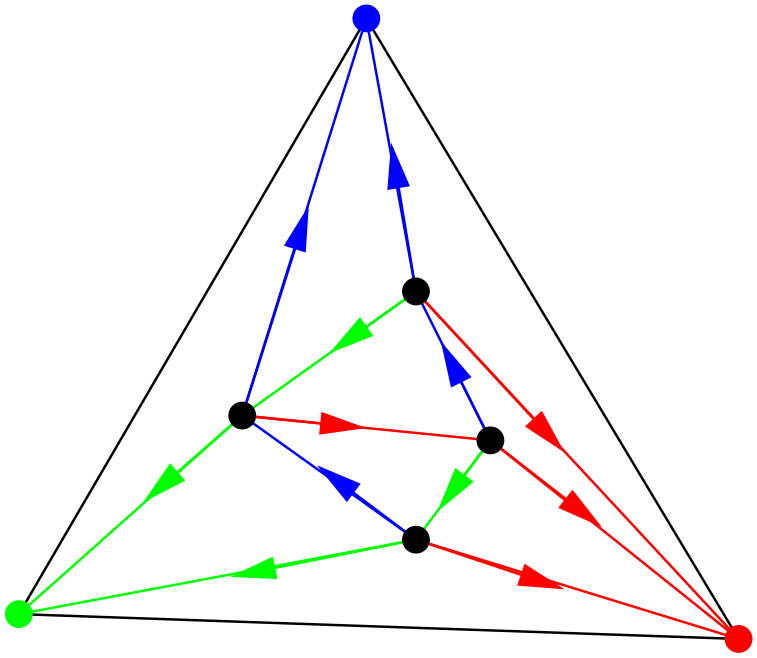
1/ We have a notion of 3-orientation for triangulations.

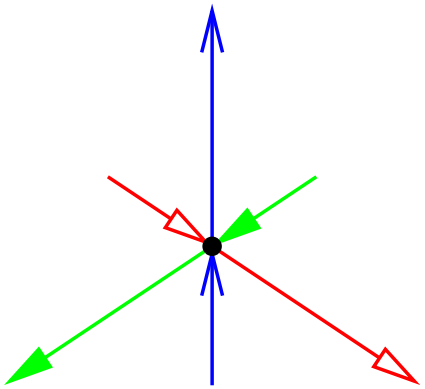


Kampen (1976)

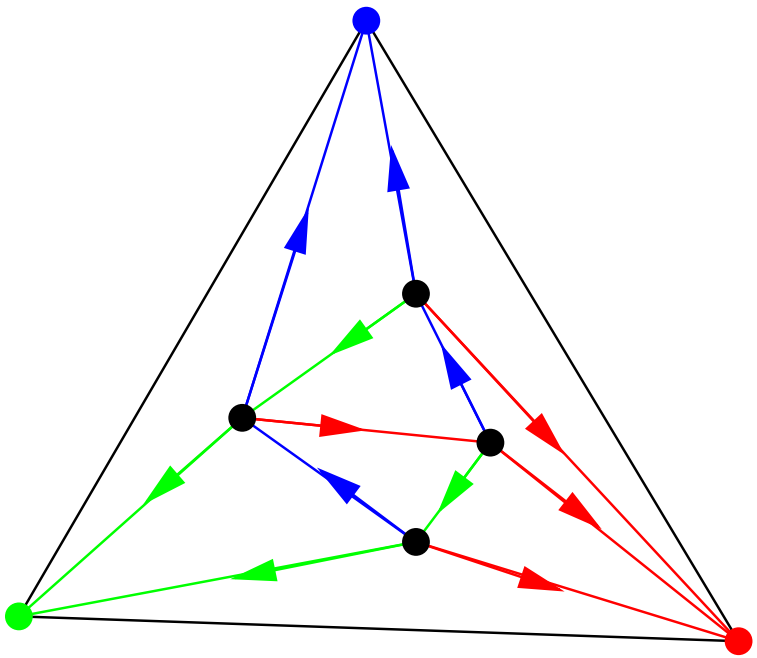
Every planar triangulation admits a 3-orientation.

2/ Every 3-orientation admits a unique Schnyder wood coloration.





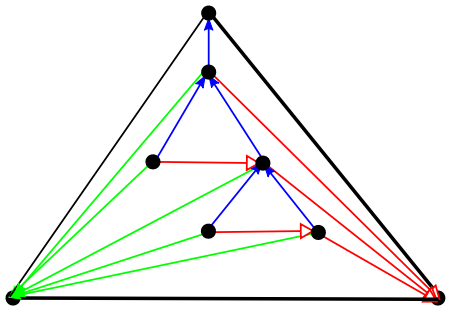
2/ Every 3-orientation admits a unique Schnyder wood coloration.

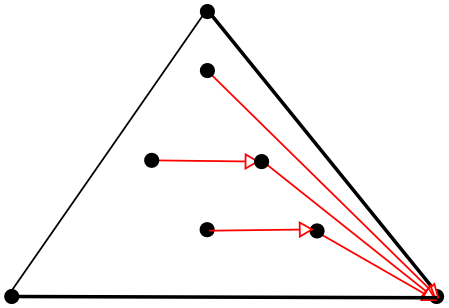


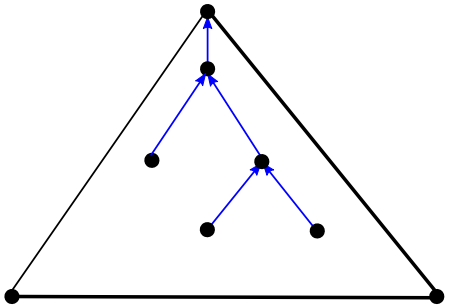
de Fraisseix and Ossona de Mendez (2001)

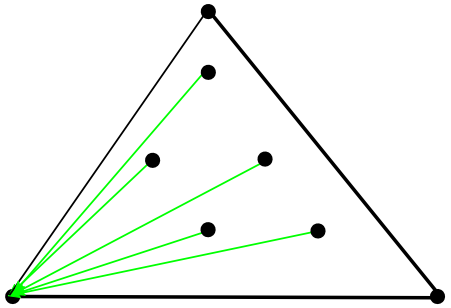
Each 3-orientation of a plane simple triangulation admits a unique coloring (up to permutation of the colors) leading to a Schnyder wood.

3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.

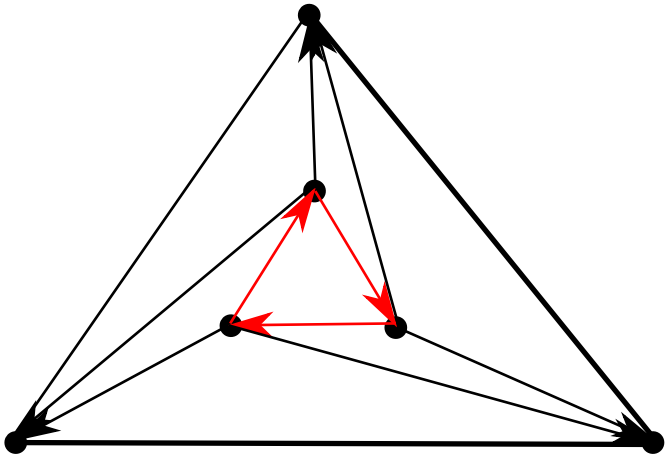


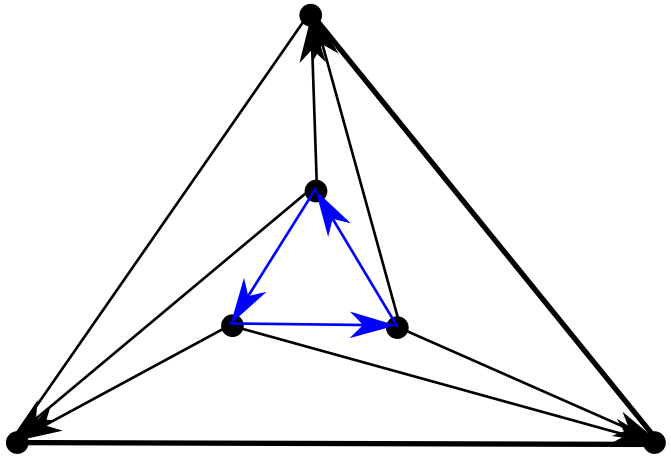






4/ The 3-orientations of a given triangulation have a structure of distributive lattice.





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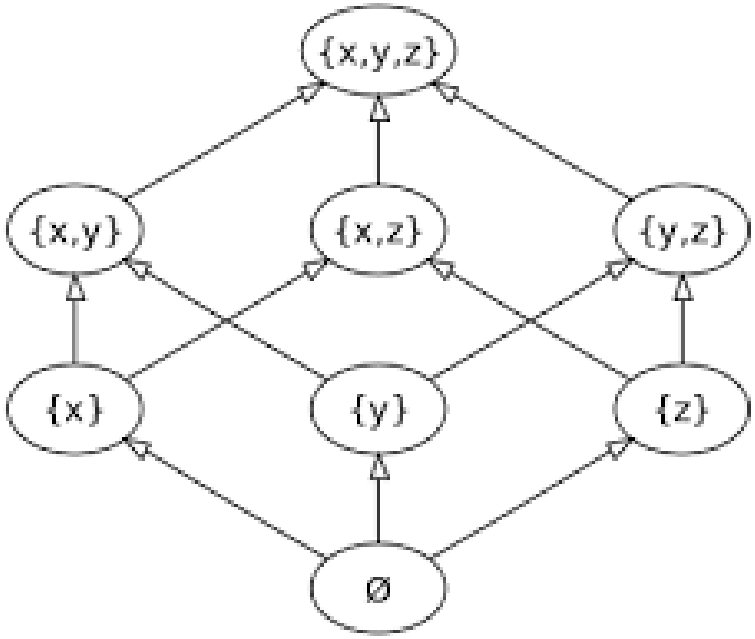

Propp (1993), Ossona de Mendez (1994), Felsner (2004)

The set of the 3-orientations of a given triangulation has a structure of distributive lattice for the appropriate ordering.

Properties of the planar case:

1/ We have a notion of 3-orientation for triangulations.

2/ Every 3-orientation admits a unique Schnyder wood coloration.

3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.

4/ The 3-orientations of a given triangulation have a structure of distributive lattice.

5/ The minimal element of the lattice has no clockwise oriented cycle.

6/ Triangulations are in bijection with a particular type of decorated embedded trees.

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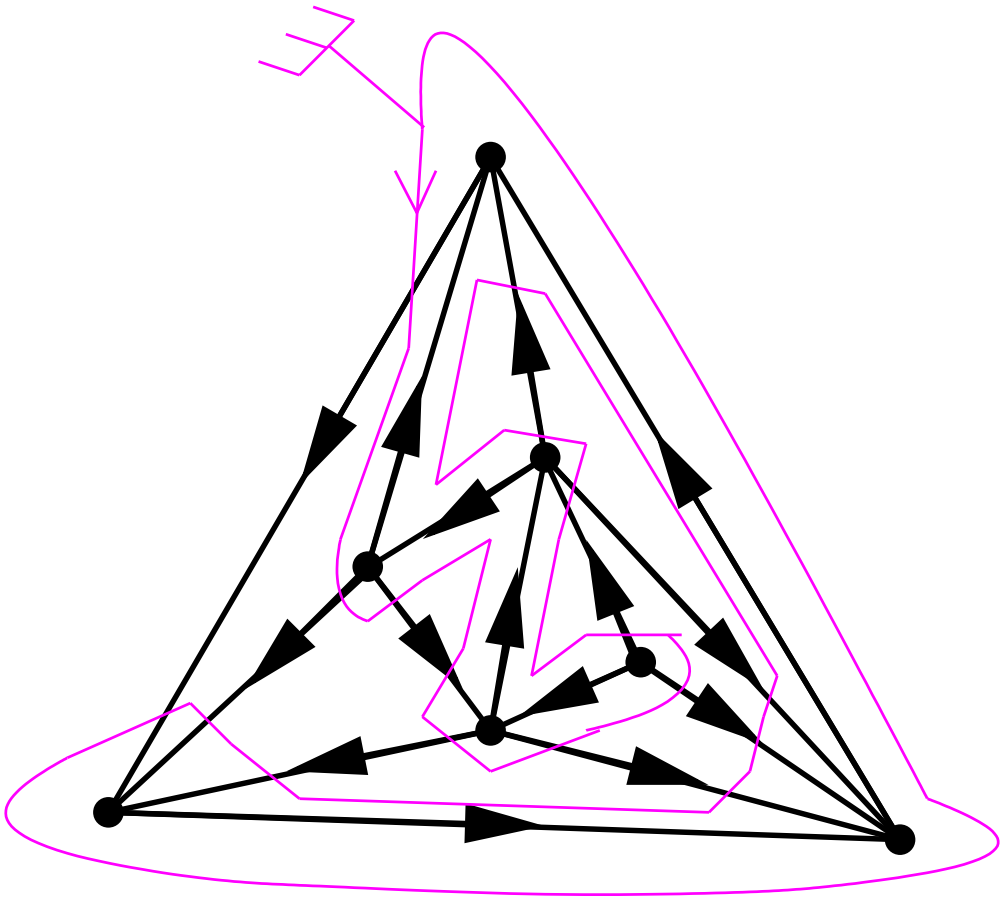

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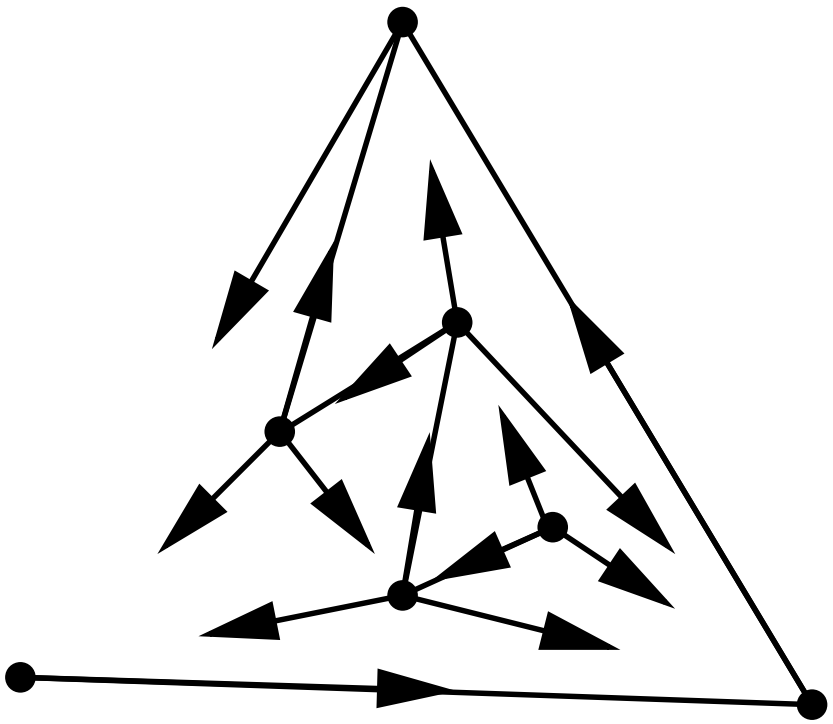

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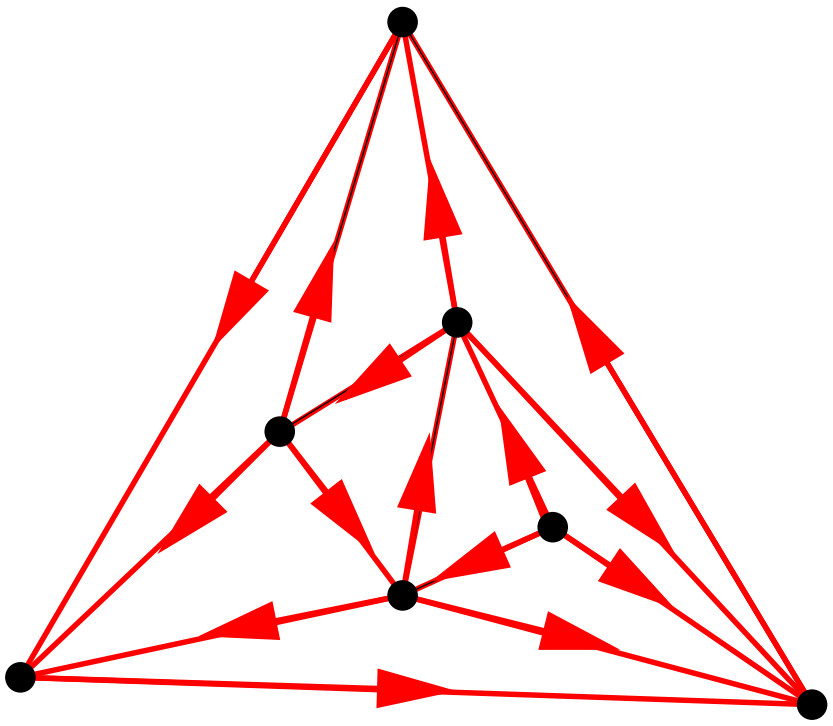

Properties of the torus case:

1/ We have a notion of 3-orientation for triangulations.

2/ Every 3-orientation admits a unique Schnyder wood coloration.

3/ ~~Each color corresponds to a spanning tree and so~~ There is no monochromatic **contractible** cycle.

4/ The 3-orientations of a given triangulation have a structure of distributive lattice**s**.

5/ The minimal element of each lattice has no clockwise oriented contractible cycle.

6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.

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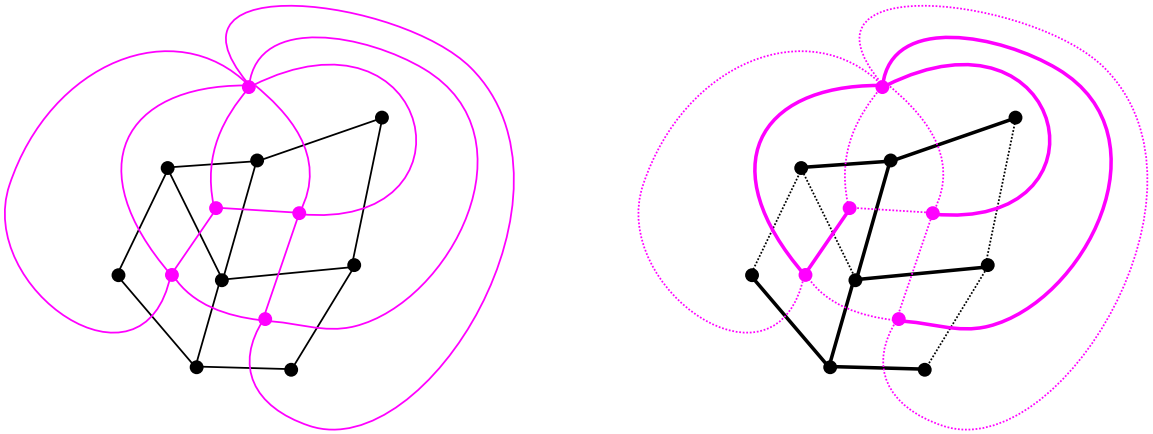
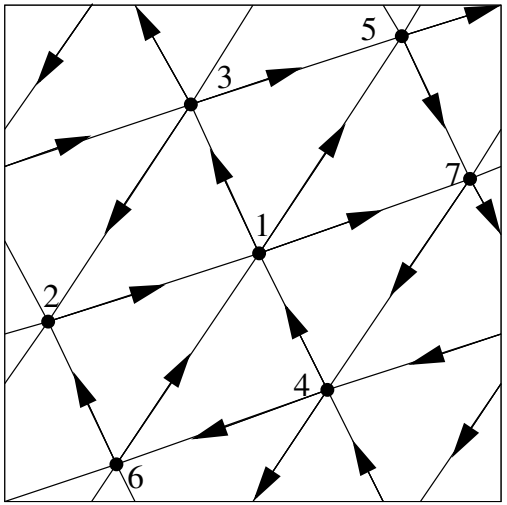
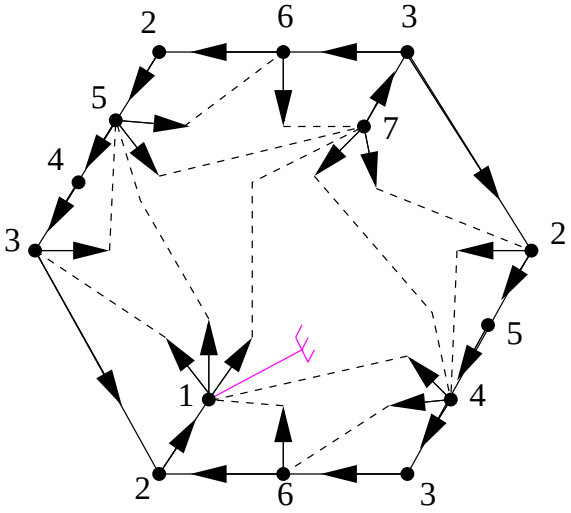


Tree-cotree Decomposition: (T, C, X). T has $n-1$ edges, C has $f-1$ edges and X the remaining.

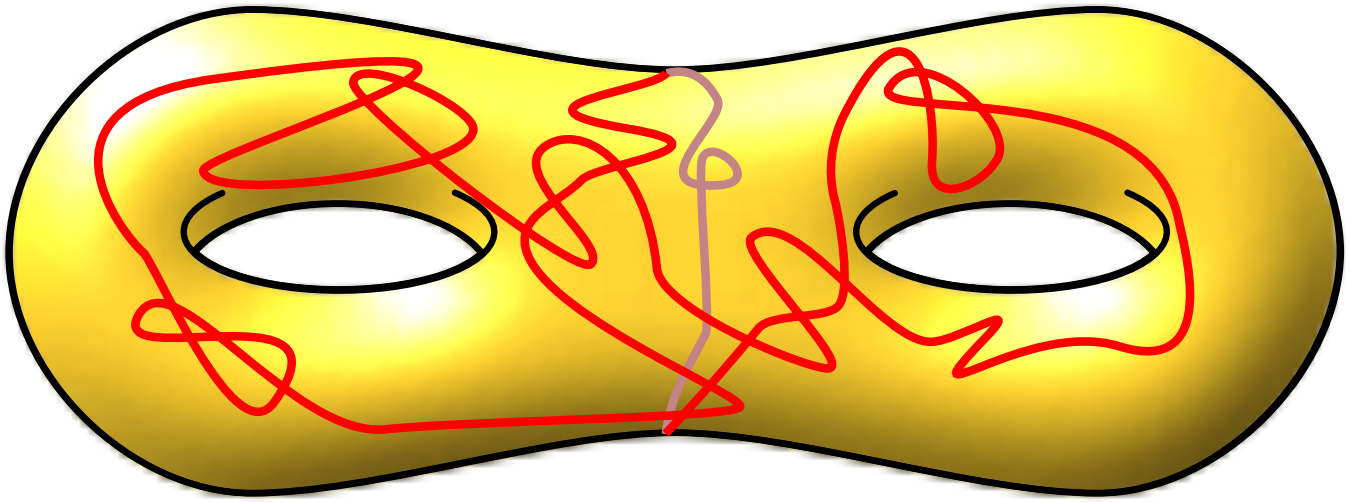
$\chi = n - (n - 1 + f - 1 + x) + f \Leftrightarrow x = 2 - \chi = 2g$

6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.





Geometric Intersection Number of Curves



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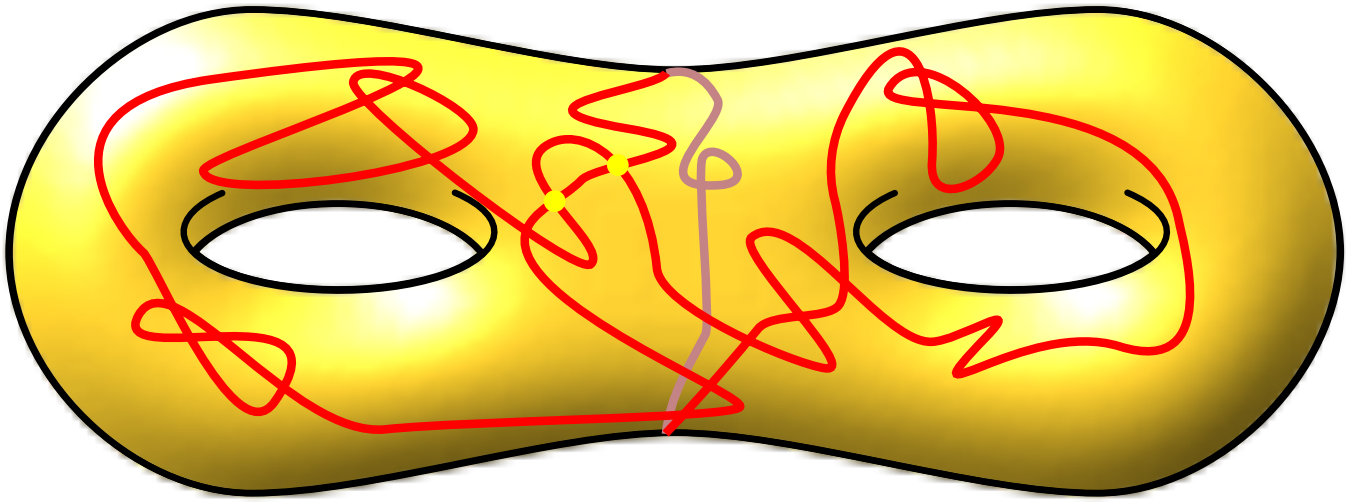

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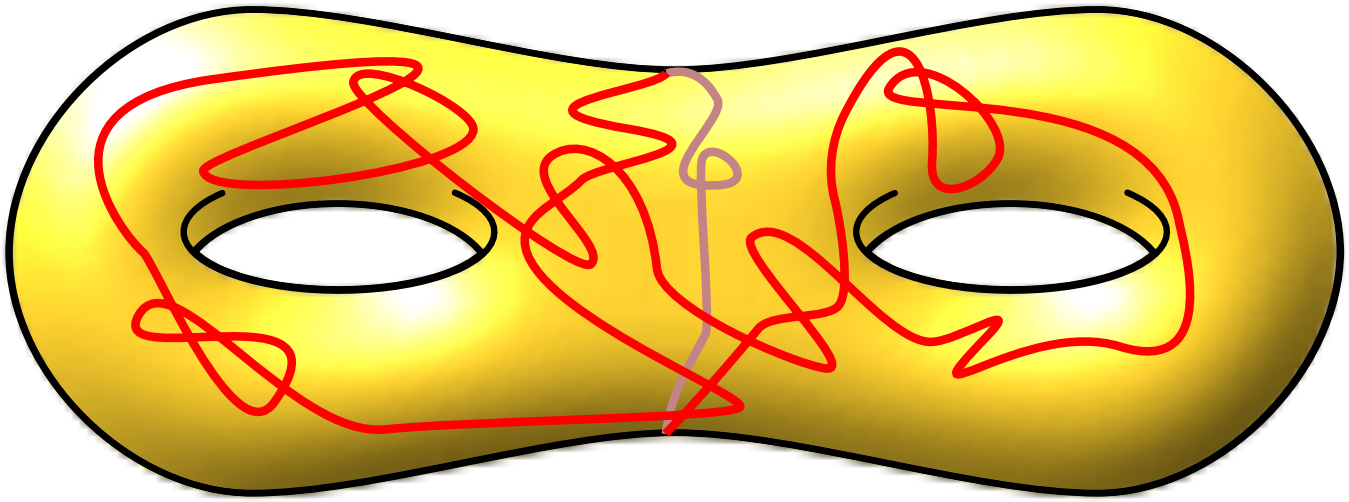
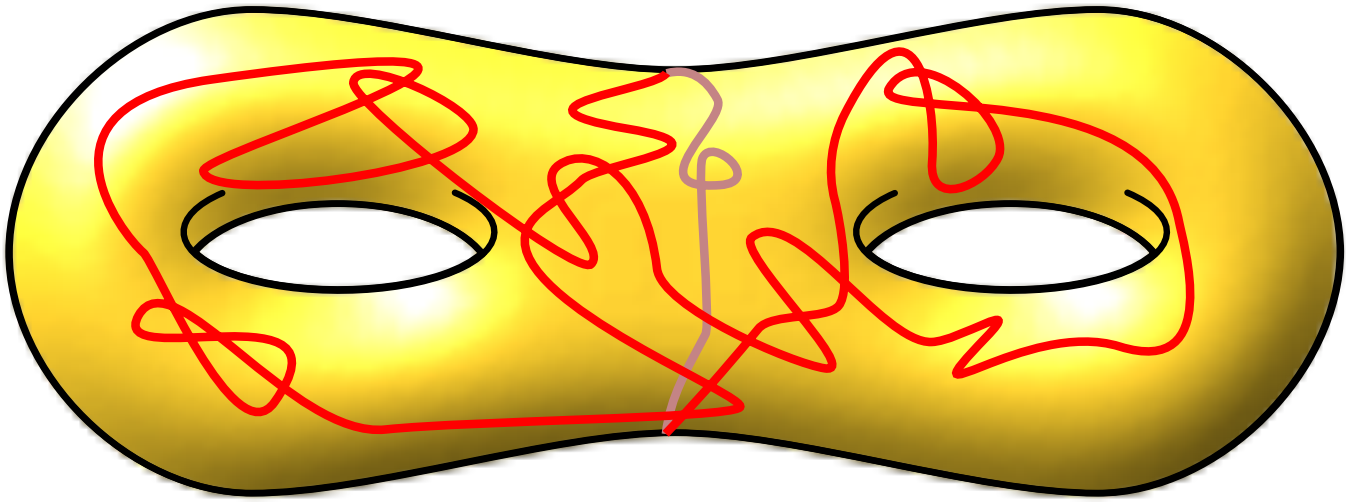
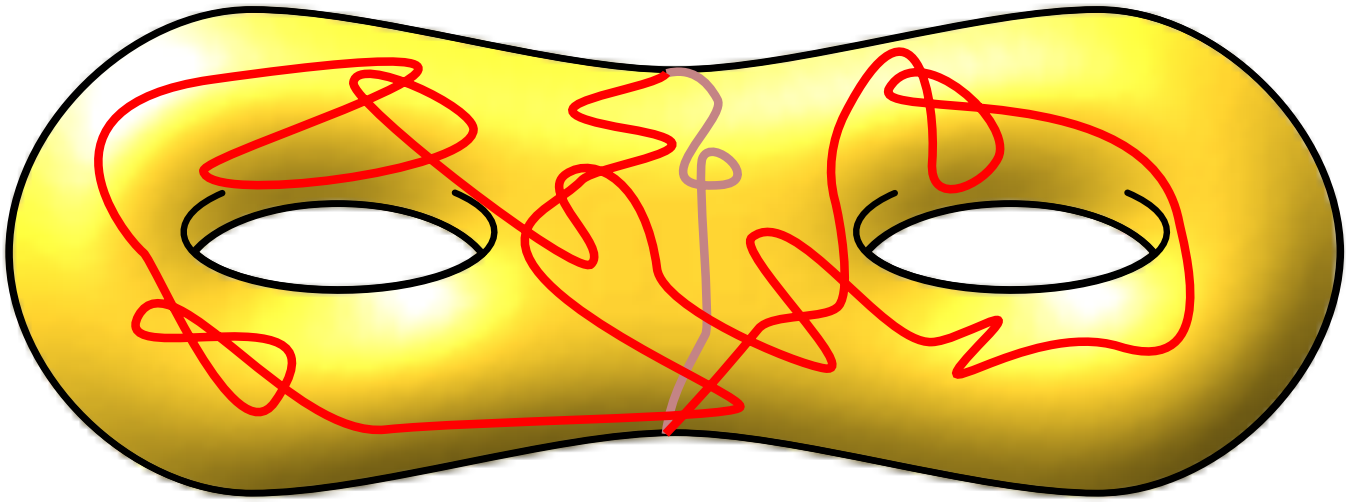


Geometric Intersection Number of Curves



Geometric Intersection Number of Curves



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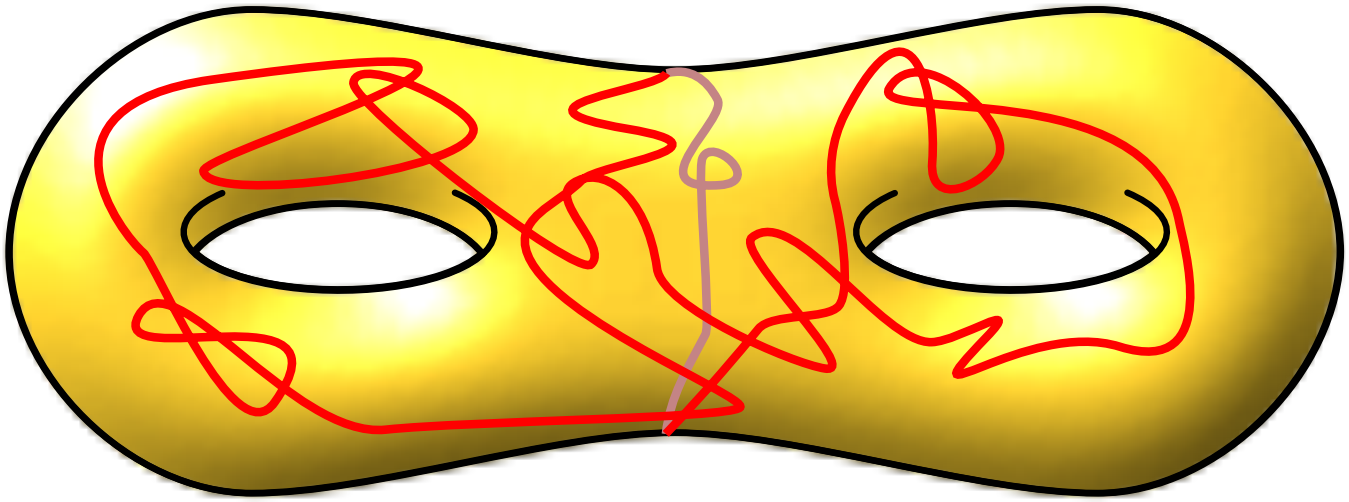

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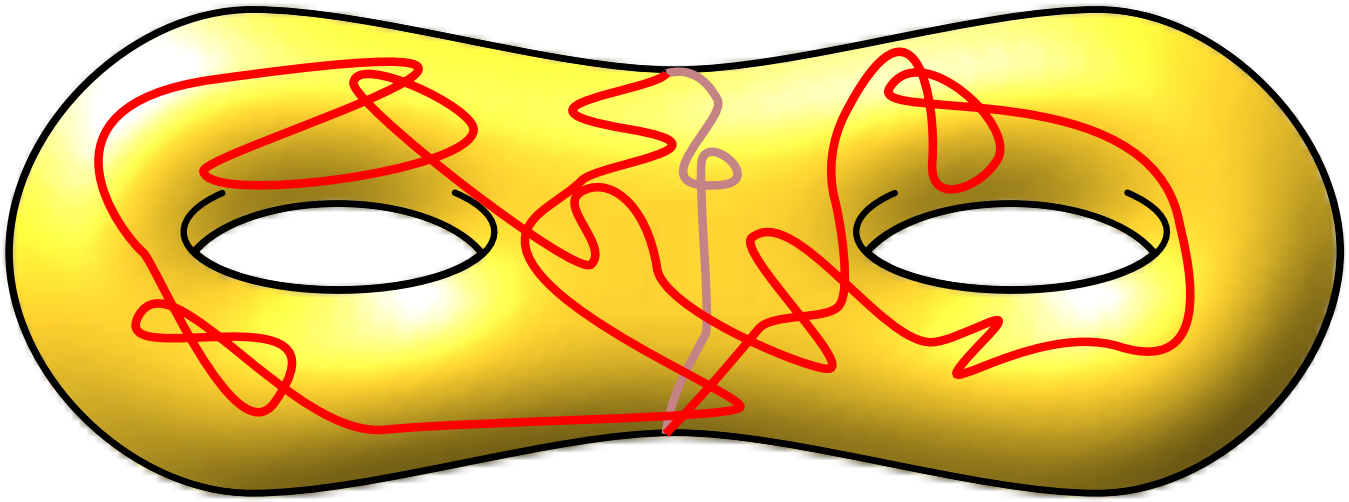

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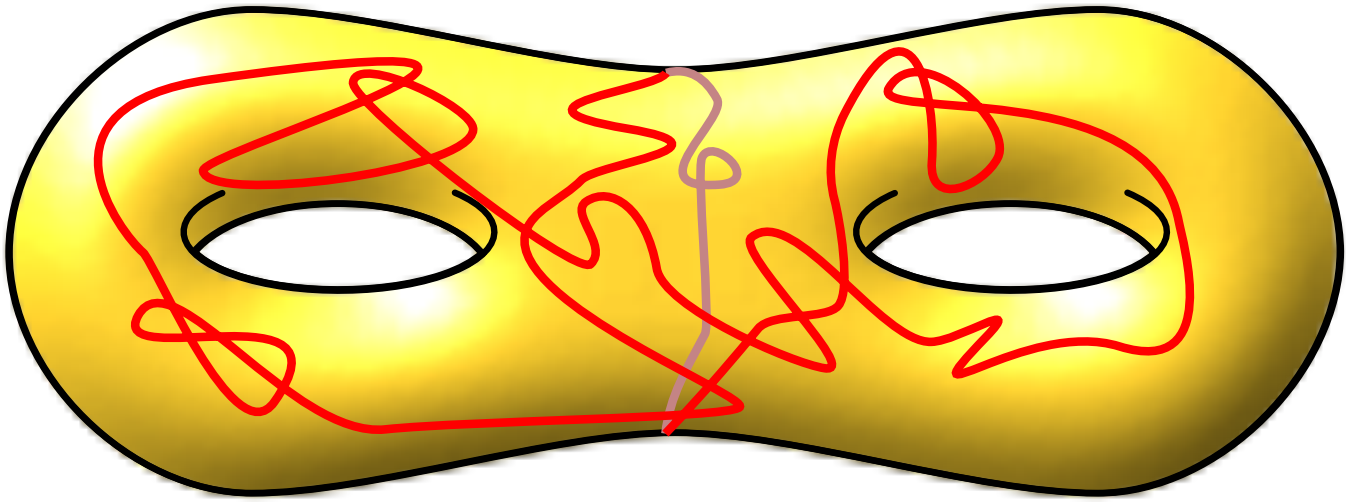

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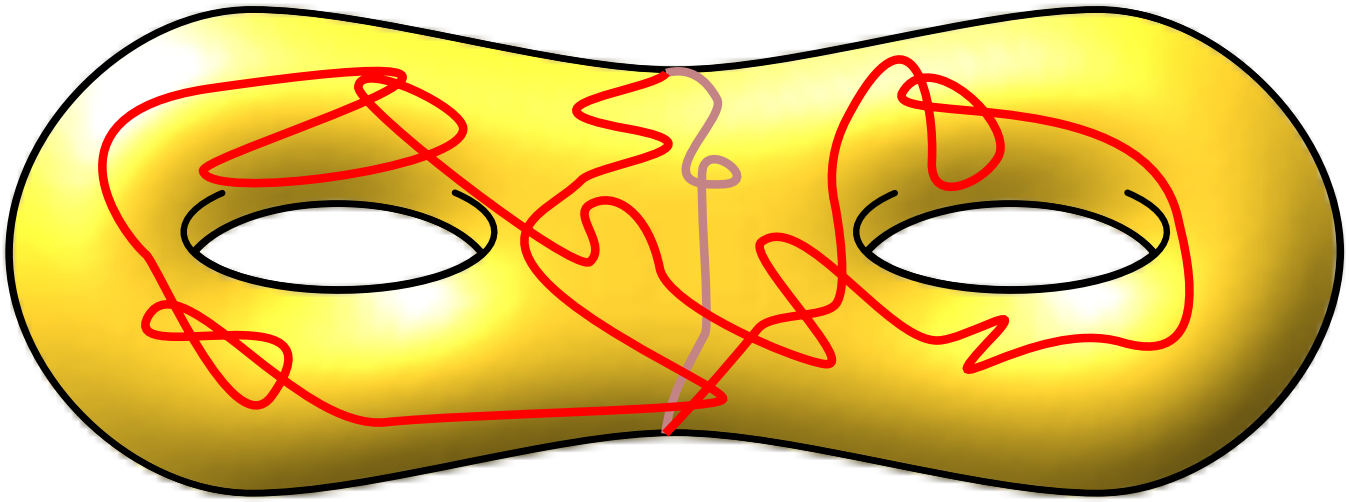

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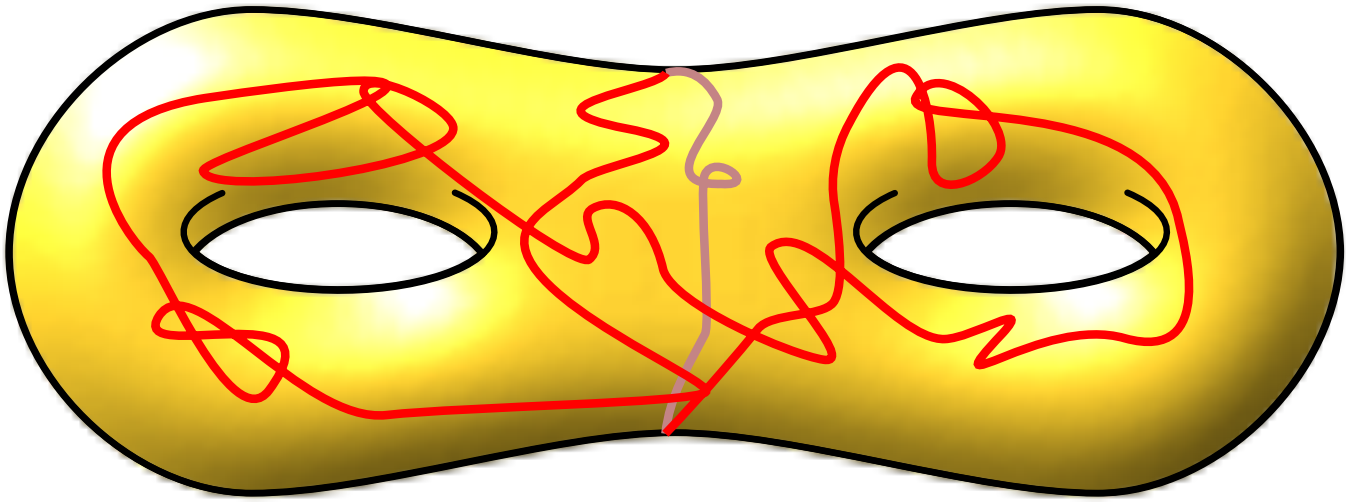

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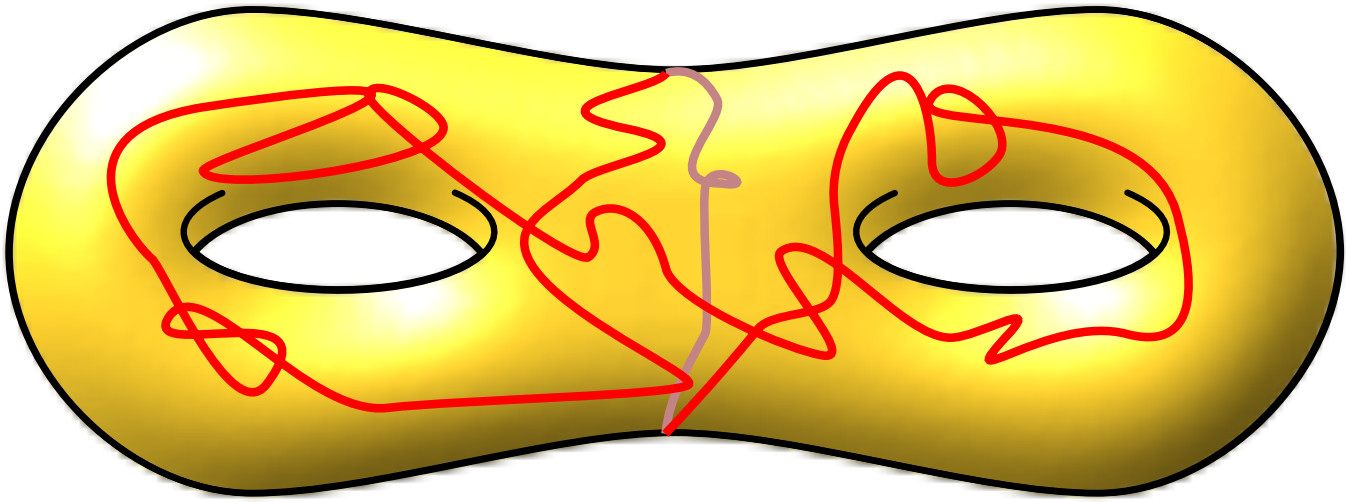

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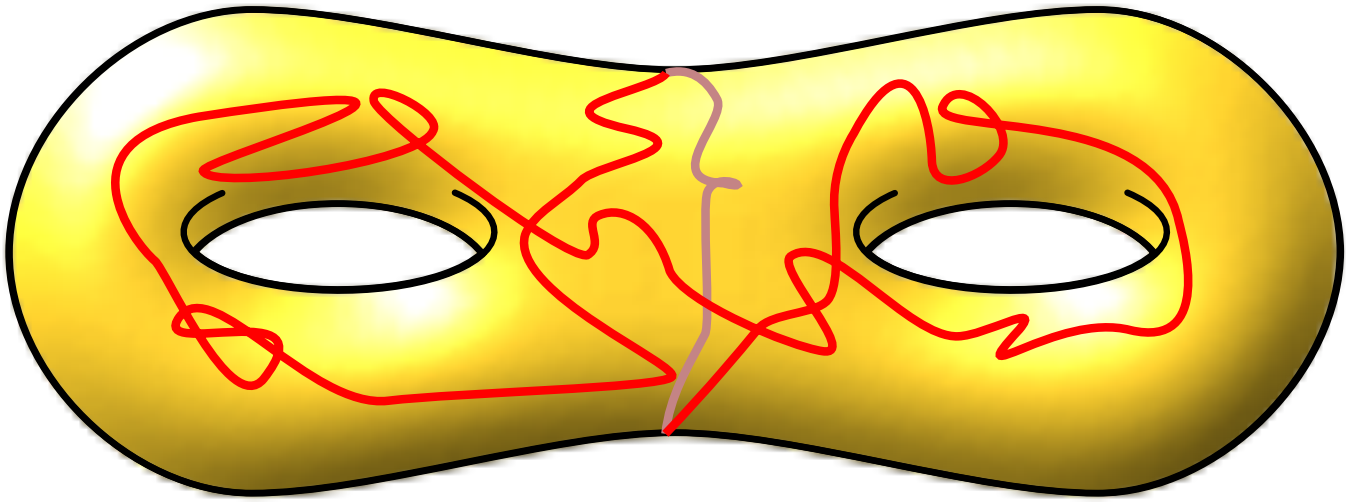

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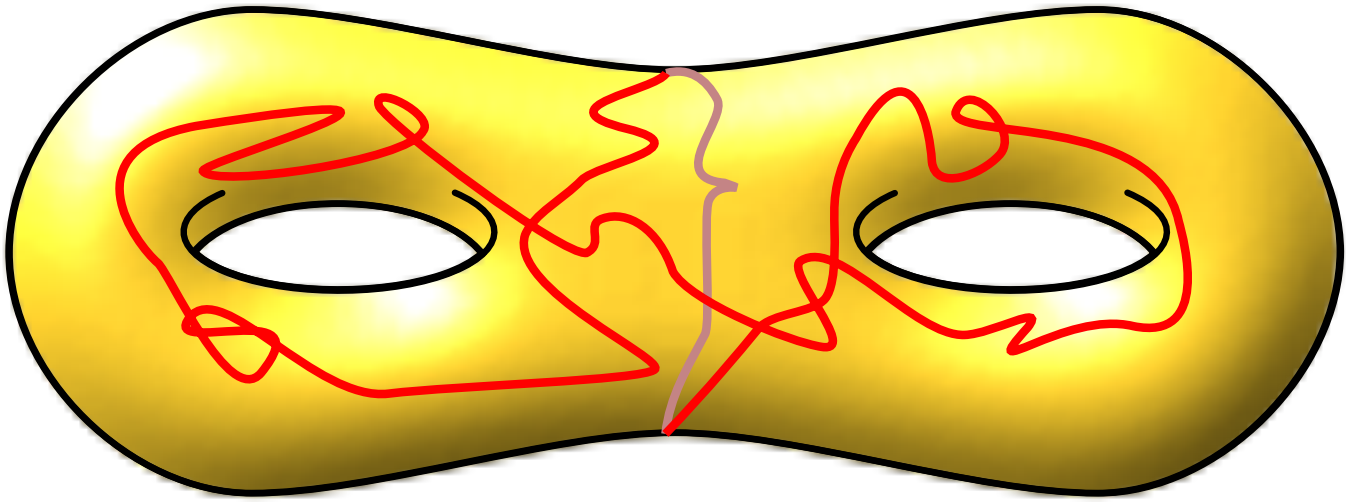

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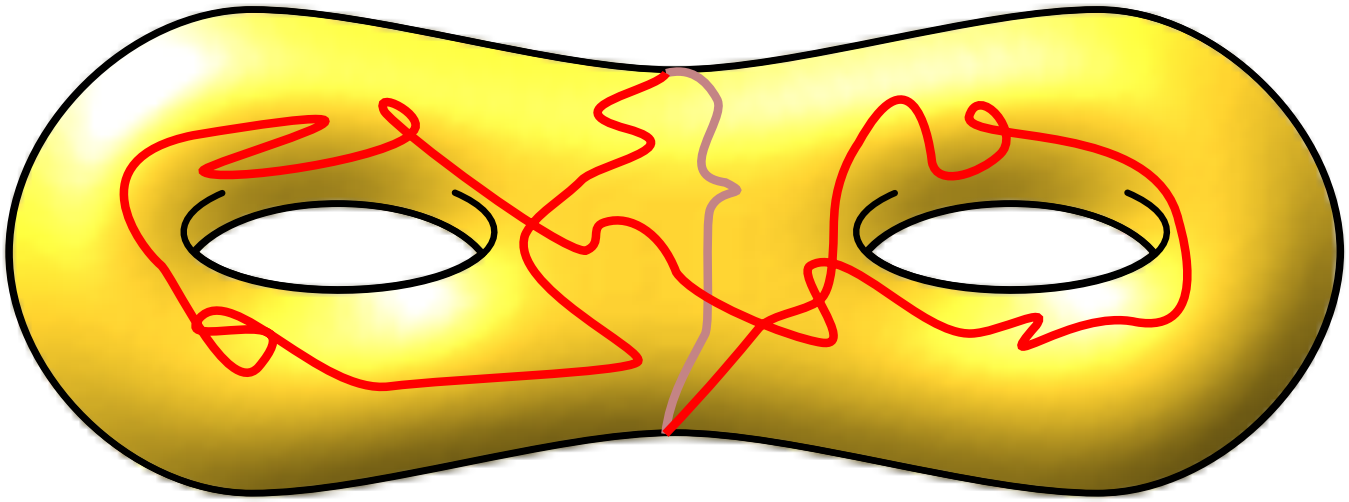

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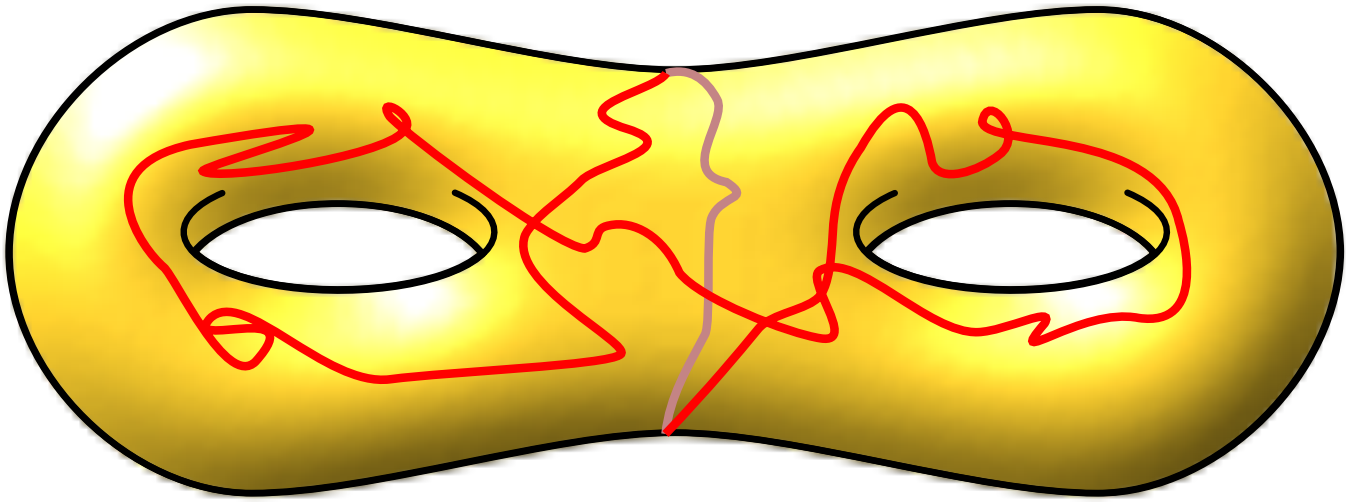

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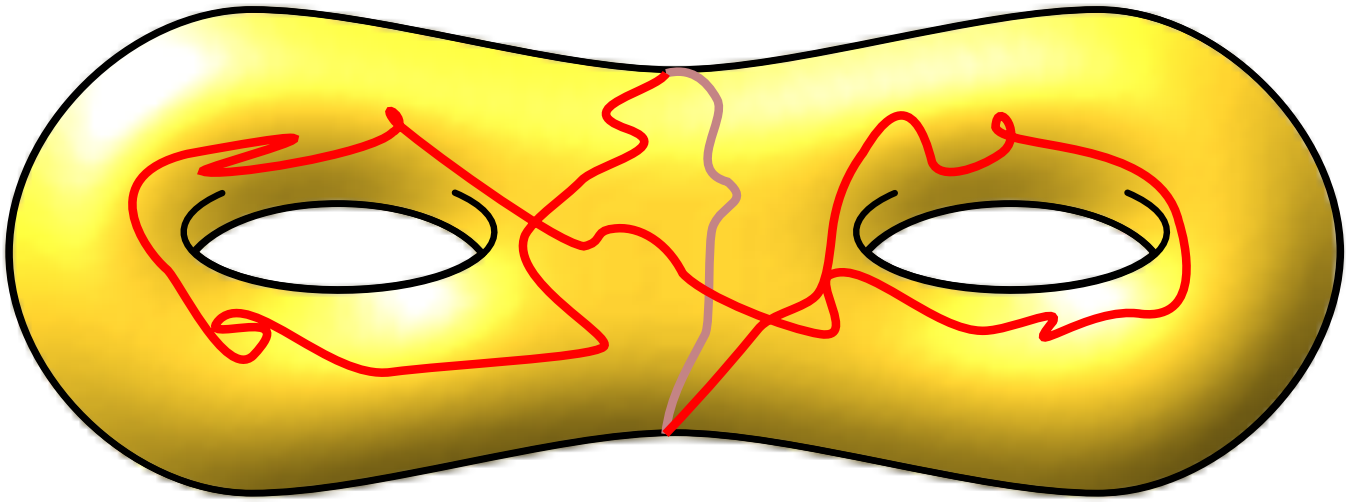

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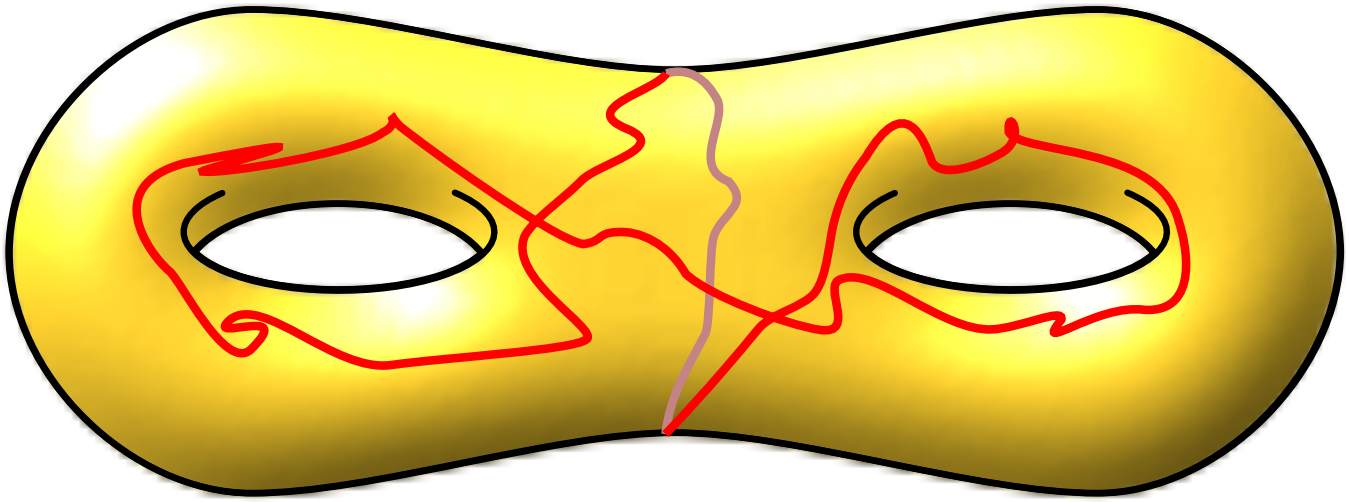

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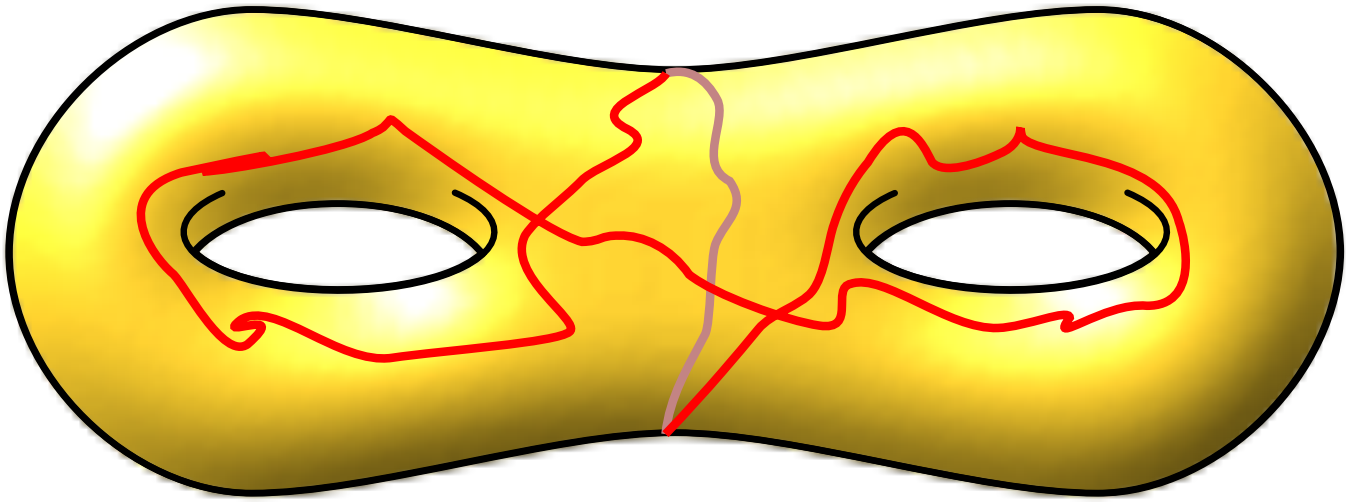

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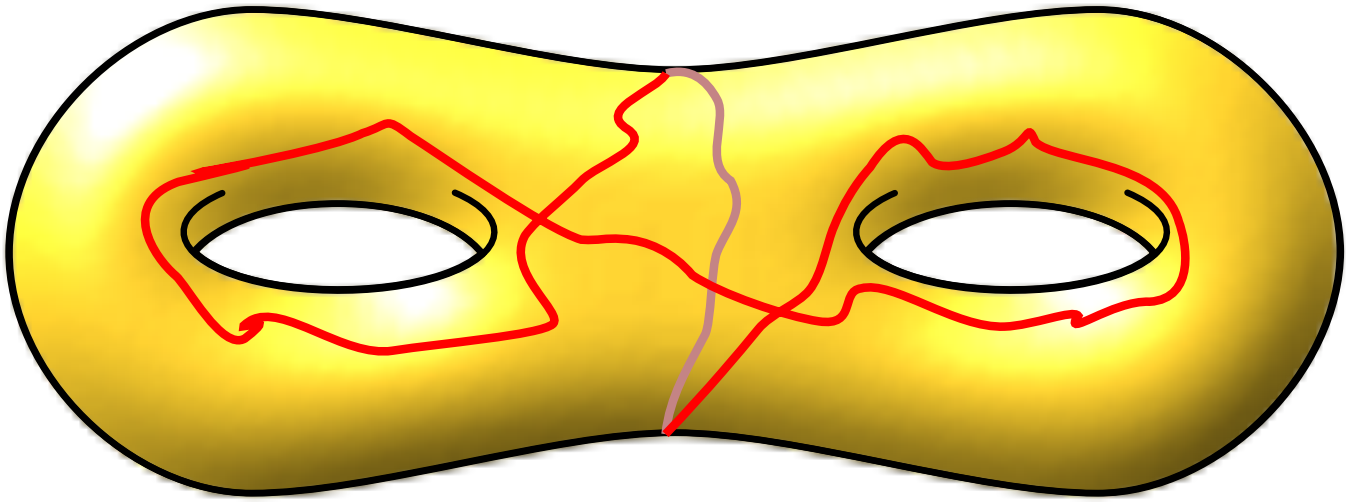

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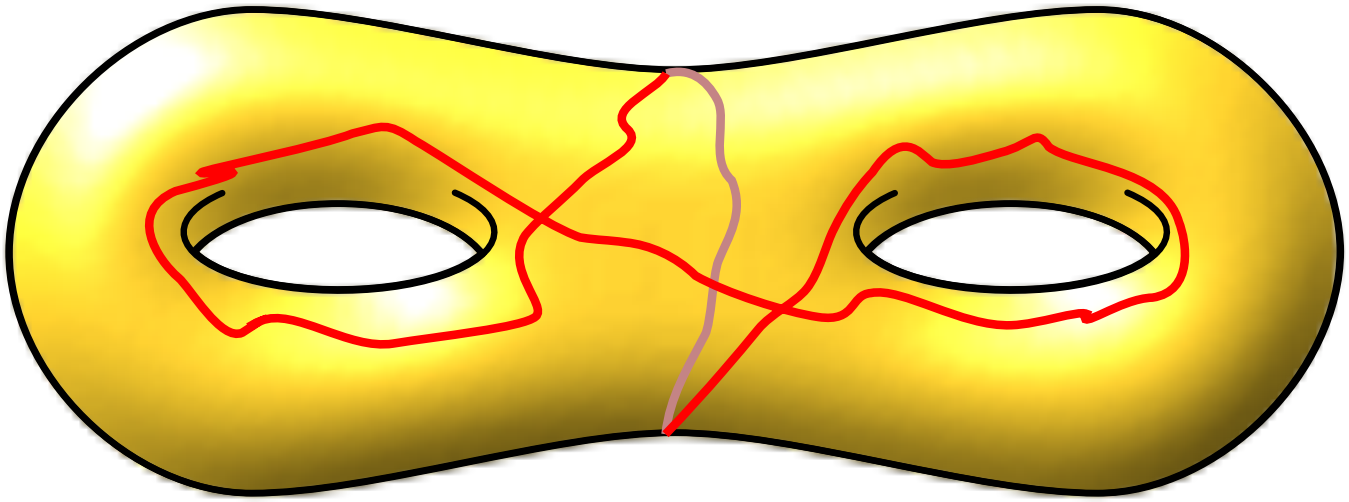

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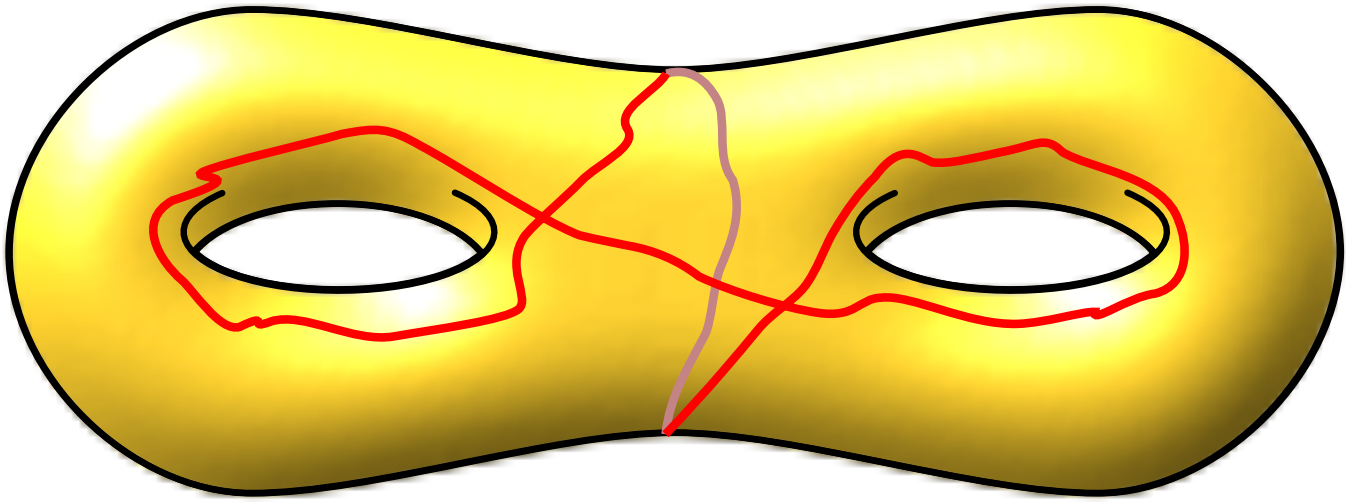

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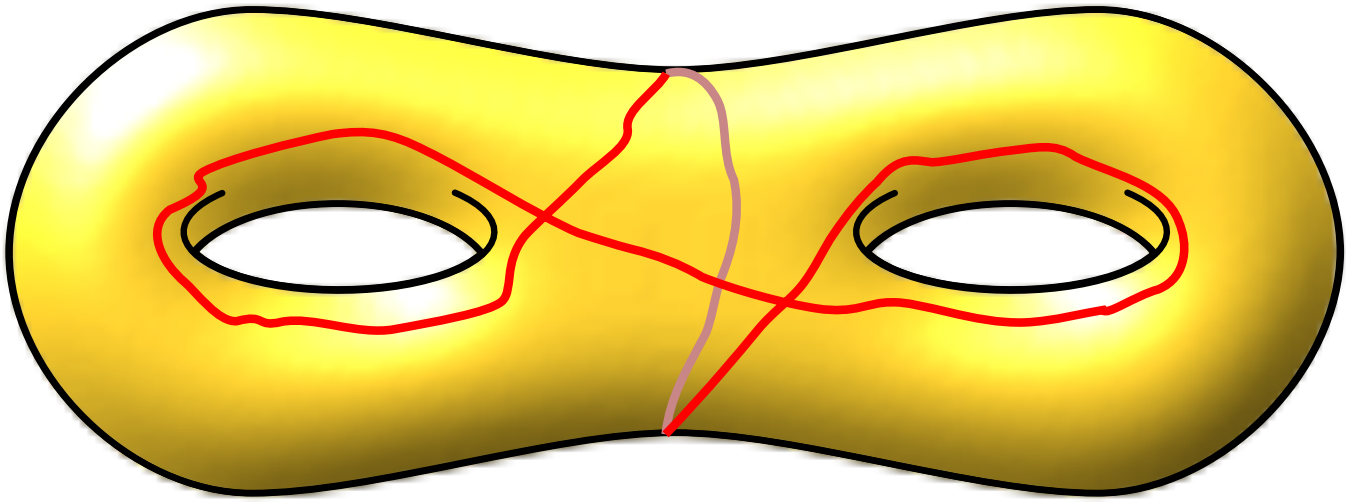

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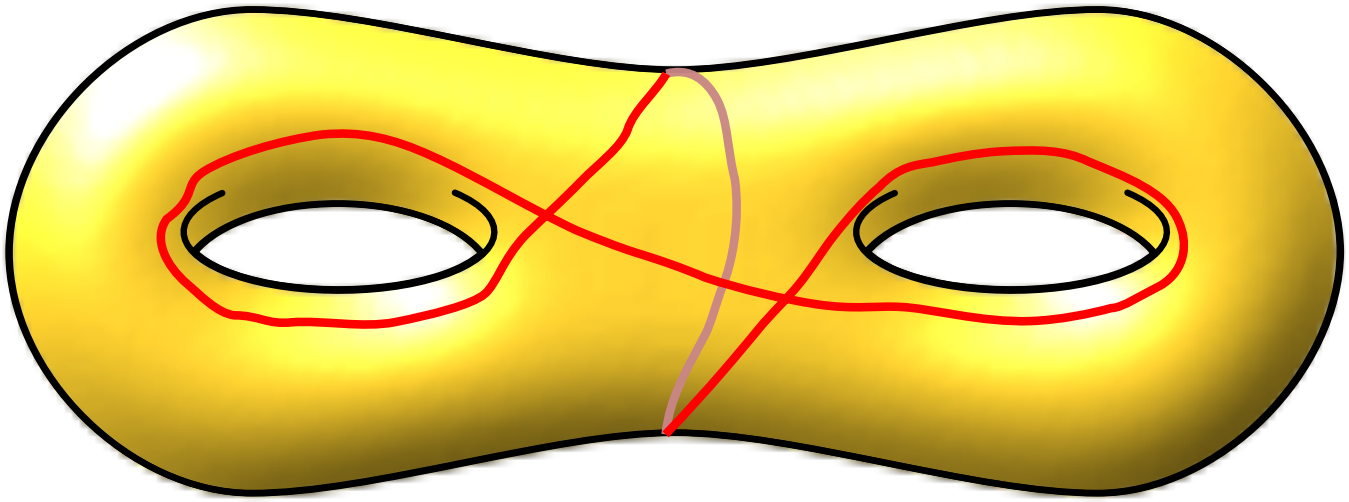

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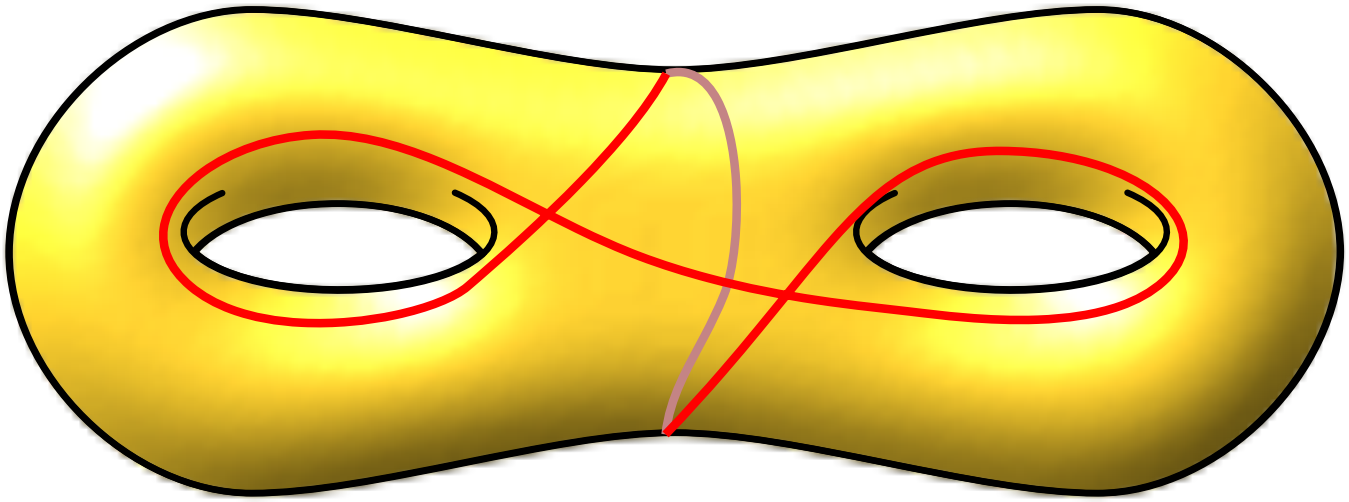

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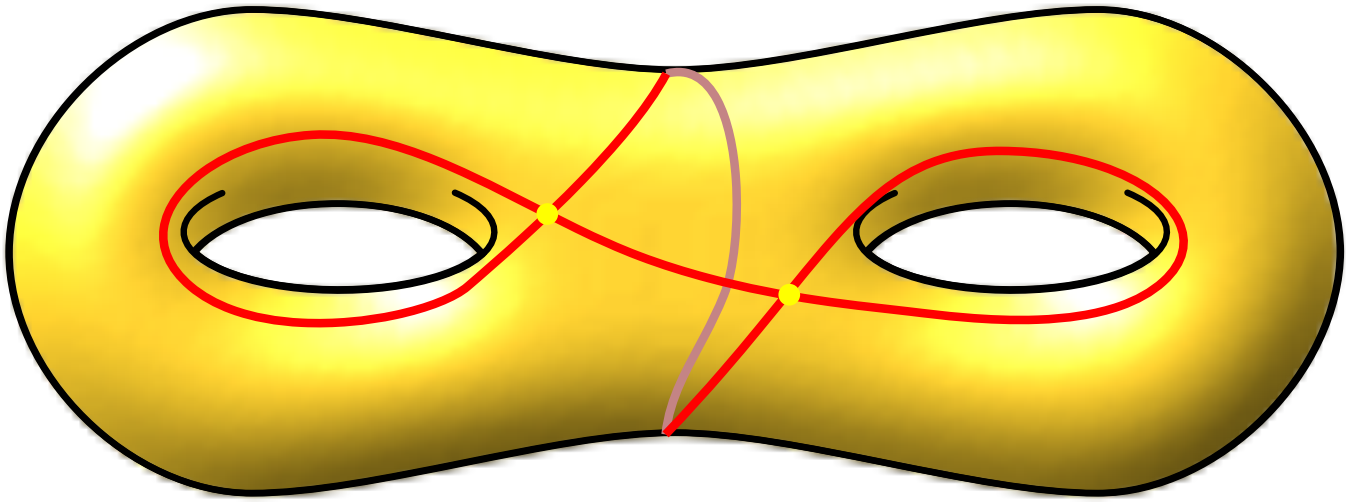
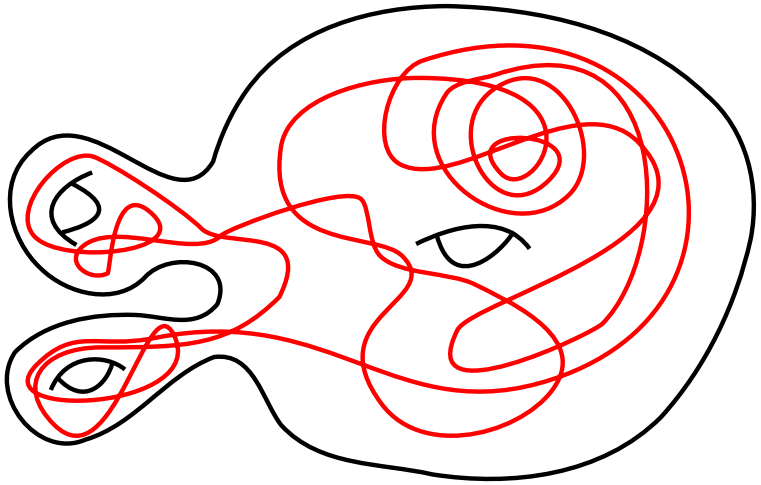
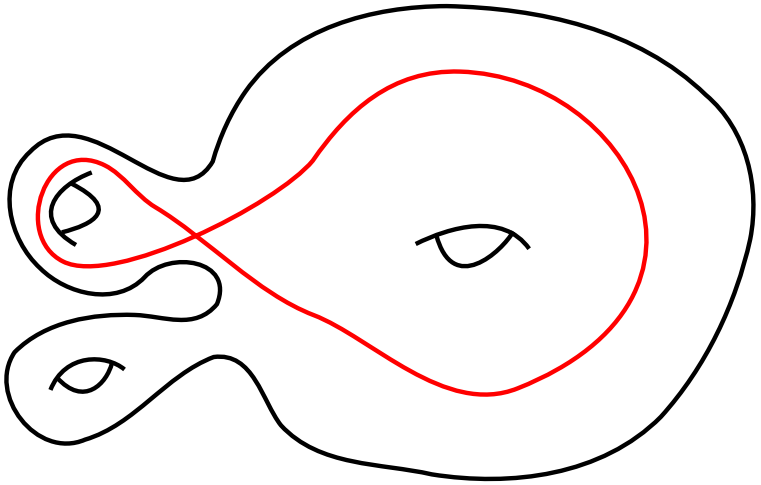




(j) Number of crossings: too many!



(k) Number of crossings: $1 \rightarrow$ optimal

Three problems:

- Deciding if a curve can be made simple by homotopy.
- Finding the minimum possible number of self-intersections.
- Finding a corresponding minimal representative.

Boundaries	Simple	Number	Representative
$b > 0$	$O((g\ell)^2)$ BS (1984)	$O((g\ell)^2)$ CL (1987)	$O((g\ell)^4)$ A (2015)
$b = 0$	? L (1987) $O(\ell^5)$ GKZ (2005)	? L (1987) $O(\ell^5)$ GKZ (2005)	? dGS (1997)
Any	$O(\ell \cdot \log^2(\ell))$	$O(\ell^2)$	$O(\ell^4)$

BS: Birman and Series, An algorithm for simple curves on surfaces.
CL: Cohen and Lustig, Paths of geodesics and geometric intersection numbers: I.
L: Lustig, Paths of geodesics and geometric intersection numbers: II.
A: Arettines, A combinatorial algorithm for visualizing representatives with minimal self-intersection.
dGS: de Graaf and Schrijver, Making curves minimally crossing by Reidemeister moves.
GKZ: Gonçalves, Kudryavtseva and Zieschang, An algorithm for minimal number of (self-)intersection points of curves on surfaces.

Publications:

1/ Some Triangulated Surfaces without Balanced Splitting: Published in *Graphs and Combinatorics*.

2/ Encoding Toroidal Triangulations: Accepted in *Discrete & Computationnal Geometry*.

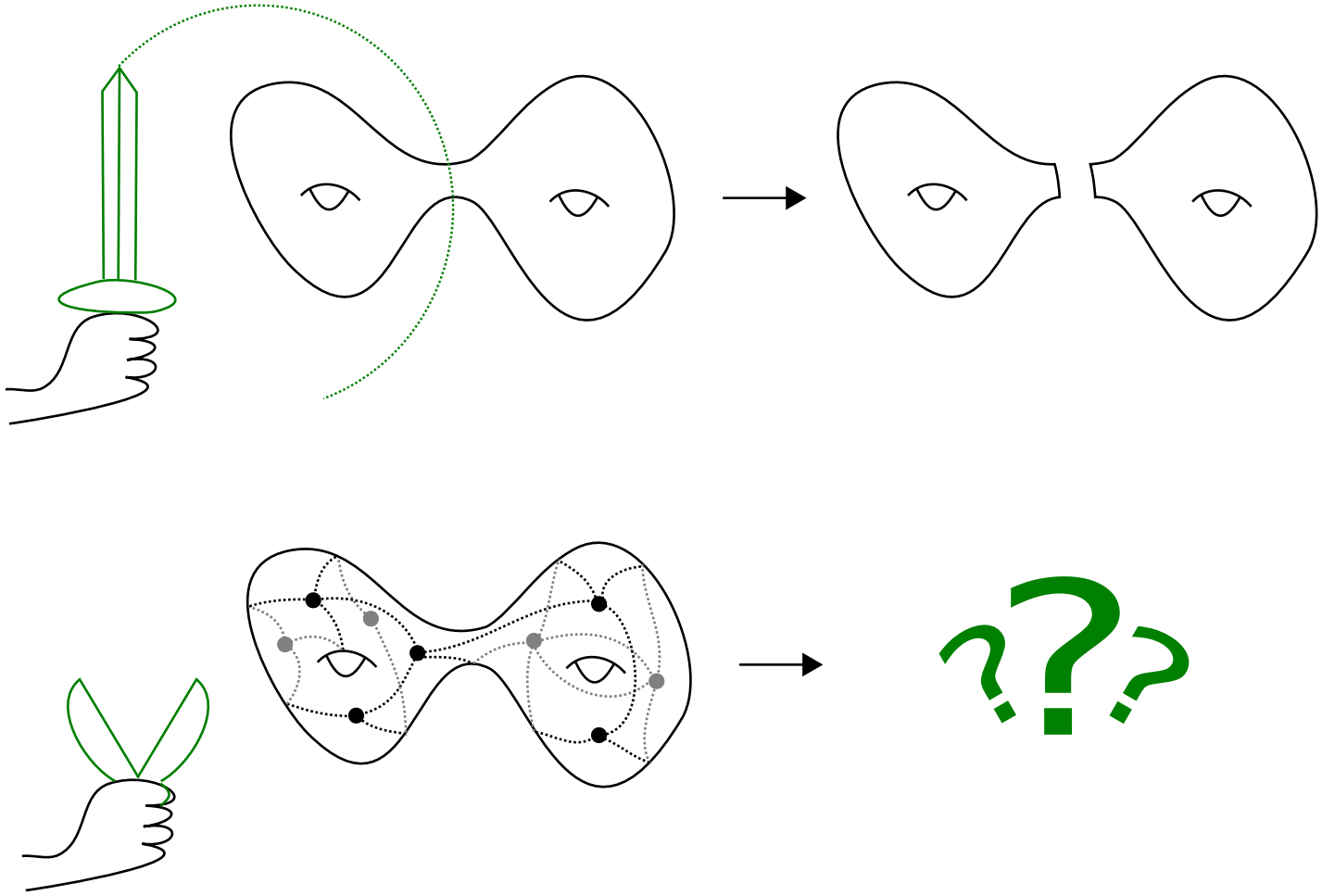
3/ Computing the Geometric Intersection Number of Curves: Will be submitted to the next SoCG.

Work in progress:

1/ Looking for a proof that does not require a computer.

2/ There are a lot of implications for the bijection in the plane. Is it possible to generalized them.

3/ It remains to look at the construction of a minimal representative for a couple of curves.



Conjecture

Deciding if there is a simple closed walk in a given homotopy class is NP-complete and FPT parametrized by the genus of the surface.

Do you have questions?