PhD Defense

Topology and Algorithms on Combinatorial Maps

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\( V \) = number of vertices, \( E \) = number of edges and \( F \) = number of faces

**Euler Formula**

On a surface that can be deformed to a sphere, any polygonal subdivision verifies:

\[
\chi(S) = V - E + F = 2
\]
Euler Formula

On a surface $S$ of genus $g$, any polygonal subdivision verifies:

$$\chi(S) = V - E + F = 2 - 2g$$
Euler Formula

On a surface $S$ of genus $g$ with $b$ boundaries, any polygonal subdivision verifies:

$$
\chi(S) = V - E + F = 2 - 2g - b
$$
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Splitting Cycles
Splitting Cycles

Deciding if a combinatorial map admits a splitting cycle is NP-complete.

Cabello et al. (2011)
Barnette’s Conjecture (1982)

Every triangulations of surfaces of genus at least 2 admit a splitting cycle.
Conjecture (Mohar and Thomassen, 2001)

Every triangulations of surfaces of genus $g \geq 2$ admit a splitting cycle of every different type.
Irreducible Triangulations

There are a finite number of irreducible triangulations of genus $g$. (Barnette and Edelson, 1988 and Joret and Wood, 2010)

- There are 396784 irreducible triangulations of genus 2.
- Unreachable for genus 3.
Genus 2 irreducible triangulations

First implementation by Thom Sulanke.

Genus 2:
Number of triangulations: 396 784

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<td>3.00</td>
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</tbody>
</table>
Genus 6

We consider the 59 non-isomorphic embeddings of $K_{12}$. (Altshuler, Bokowski and Schuchert 1996)

Average: 7.58  
Worst-case: 8

Average: 9.41  
Worst-case: 10

Average: 10.32  
Worst-case: 12 (Hamiltonian cycle!)
Complete Graphs

\[ \chi(S) = v - e + f = n - \frac{n(n - 1)}{2} + \frac{2}{3} \cdot \frac{n(n - 1)}{2} = 2 - 2g \]

\[ g = \frac{(n - 3)(n - 4)}{12} \]

\[ (n - 3)(n - 4) \equiv 0[12] \iff n \equiv 0, 3, 4 \text{ or } 7[12] \]

**Theorem (Ringel and Youngs, \sim 1970)**

\( K_n \) can triangulate a surface if and only if \( n \equiv 0, 3, 4 \text{ or } 7[12] \).
Computation time

New implementation in C++. The data-structure used for the triangulations is the flag representation.

<table>
<thead>
<tr>
<th>n</th>
<th>12</th>
<th>15</th>
<th>16</th>
<th>19</th>
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<tr>
<td>basic</td>
<td>2 s.</td>
<td>1 h.</td>
<td>12 h.</td>
<td>~10 years</td>
</tr>
</tbody>
</table>

This has been computed with an 8 cores computer with 16 Go of RAM. It uses parallel computation.
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<tr>
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<td>3 s.</td>
<td>8 sec.</td>
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<td>1 h.</td>
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\[\downarrow = \text{No cycle found.}\]

#### Counter-Examples

Mohar and Thomassen conjecture is false.
<table>
<thead>
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<th>Type</th>
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<th>$K_{27}$</th>
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</tbody>
</table>

⊥ = No cycle found.

**Conjecture**

For every $\alpha > 0$, there exists a triangulation with no splitting cycles of type larger than $\alpha \cdot \frac{g}{2}$. 
Encoding Toroidal Triangulations

Properties of the planar case:

1/ We have a notion of 3-orientation for triangulations.
2/ Every 3-orientation admits a unique Schnyder wood coloration.
3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.
4/ The 3-orientations of a given triangulation have a structure of distributive lattice.
5/ The minimal element of the lattice has no clockwise oriented cycle.
6/ Triangulations are in bijection with a particular type of decorated embedded trees.
1/ We have a notion of 3-orientation for triangulations.

Kampen (1976)

Every planar triangulation admits a 3-orientation.
2/ Every 3-orientation admits a unique Schnyder wood coloration.
2/ Every 3-orientation admits a unique Schnyder wood coloration.

de Fraisseix and Ossona de Mendez (2001)

Each 3-orientation of a plane simple triangulation admits a unique coloring (up to permutation of the colors) leading to a Schnyder wood.
3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.
4/ The 3-orientations of a given triangulation have a structure of distributive lattice.
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The set of the 3-orientations of a given triangulation has a structure of distributive lattice for the appropriate ordering.
Properties of the planar case:

1/ We have a notion of 3-orientation for triangulations.

2/ Every 3-orientation admits a unique Schnyder wood coloration.

3/ Each color corresponds to a spanning tree and so there is no monochromatic cycle.

4/ The 3-orientations of a given triangulation have a structure of distributive lattice.

5/ The minimal element of the lattice has no clockwise oriented cycle.

6/ Triangulations are in bijection with a particular type of decorated embedded trees.
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Properties of the torus case:

1/ We have a notion of 3-orientation for triangulations.
2/ Every 3-orientation admits a unique Schnyder wood coloration.
3/ Each color corresponds to a spanning tree and so There is no monochromatic contractible cycle.
4/ The 3-orientations of a given triangulation have a structure of distributive lattices.
5/ The minimal element of each lattice has no clockwise oriented contractible cycle.
6/ Triangulations are in bijection with a particular type of decorated unicellular toroidal maps.
6/ Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps.**

Tree-cotree Decomposition: \((T, C, X)\). \(T\) has \(n - 1\) edges, \(C\) has \(f - 1\) edges and \(X\) the remaining.

\[
\chi = n - (n - 1 + f - 1 + x) + f \iff x = 2 - \chi = 2g
\]
6. Triangulations are in bijection with a particular type of decorated **unicellular toroidal maps**.
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Three problems:

- Deciding if a curve can be made simple by homotopy.
- Finding the minimum possible number of self-intersections.
- Finding a corresponding minimal representative.

(j) Number of crossings: too many!

(k) Number of crossings: 1 → optimal
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<tr>
<th>Boundaries</th>
<th>Simple</th>
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<th>Representative</th>
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<td>$O((g\ell)^2)$</td>
<td>$O((g\ell)^4)$</td>
</tr>
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<td>$b = 0$</td>
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<td>$O(\ell^5)$</td>
<td>$dGS (1997)$</td>
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<td>Any</td>
<td>$O(\ell \cdot \log^2(\ell))$</td>
<td>$O(\ell^2)$</td>
<td>$O(\ell^4)$</td>
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</tbody>
</table>

**BS:** Birman and Series, An algorithm for simple curves on surfaces.
**CL:** Cohen and Lustig, Paths of geodesics and geometric intersection numbers: I.
**L:** Lustig, Paths of geodesics and geometric intersection numbers: II.
**A:** Arettines, A combinatorial algorithm for visualizing representatives with minimal self-intersection.
**dGS:** de Graaf and Schrijver, Making curves minimally crossing by Reidemeister moves.
**GKZ:** Gonçalves, Kudryavtseva and Zieschang, An algorithm for minimal number of (self-)intersection points of curves on surfaces.
Publications:

1/ Some Triangulated Surfaces without Balanced Splitting: Published in *Graphs and Combinatorics*.

2/ Encoding Toroidal Triangulations: Accepted in *Discrete & Computationnal Geometry*.

3/ Computing the Geometric Intersection Number of Curves: Will be submitted to the next SoCG.

Work in progress:

1/ Looking for a proof that does not require a computer.

2/ There are a lot of implications for the bijection in the plane. Is it possible to generalized them.

3/ It remains to look at the construction of a minimal representative for a couple of curves.
Conjecture

Deciding if there is a simple closed walk in a given homotopy class is NP-complete and FPT parametrized by the genus of the surface.
Do you have questions?