Formal Verification of Cyber-Physical Systems

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Cyber-Physical Systems Summer School

EIT ICT Labs - PERSYVAL Lab

Grenoble, France

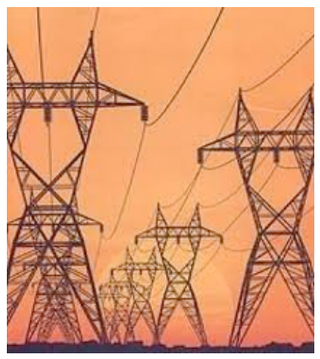
July 09, 2013

Cyber-Physical Systems

Computation

Control

Communication



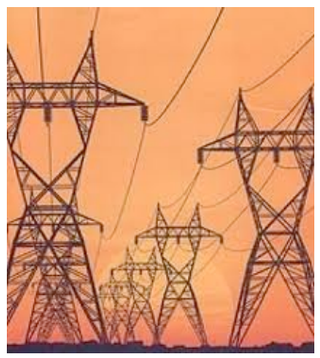
Cyber-Physical Systems

Computation

Control

Communication

Safety-criticality



Recalls due to Software Bugs

February 6, 2010:

Toyota recalls 133,000 Prius vehicles in the US and 52, 000 in Europe to fix problems with its anti-lock brake software

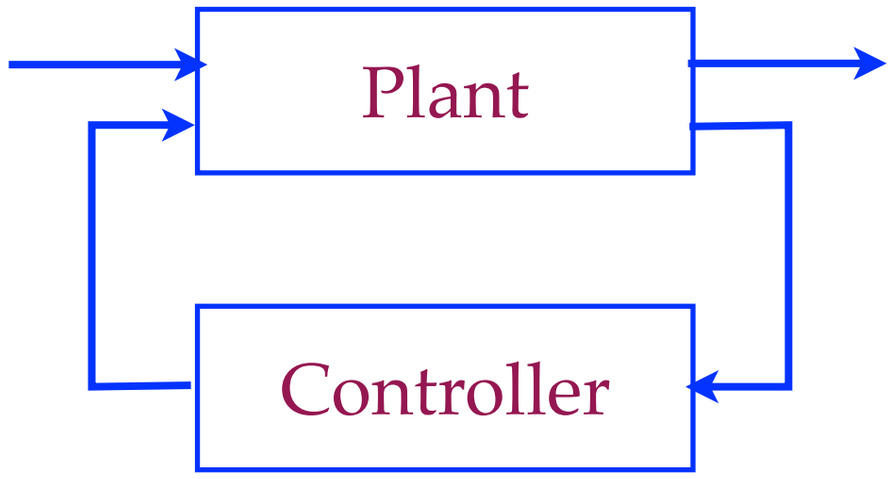
1990-2000:

200,000 devices affected due to safety recalls of pacemakers and implantable cardioverter defibrillators due to firmware problems.

Grand Challenge: Development of high-confidence Cyber-Physical Systems

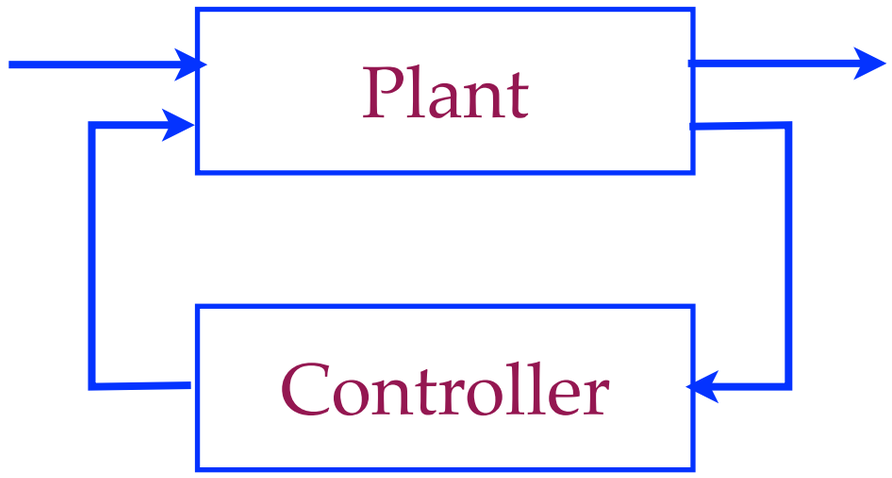
Model-based Design

Model-based Design



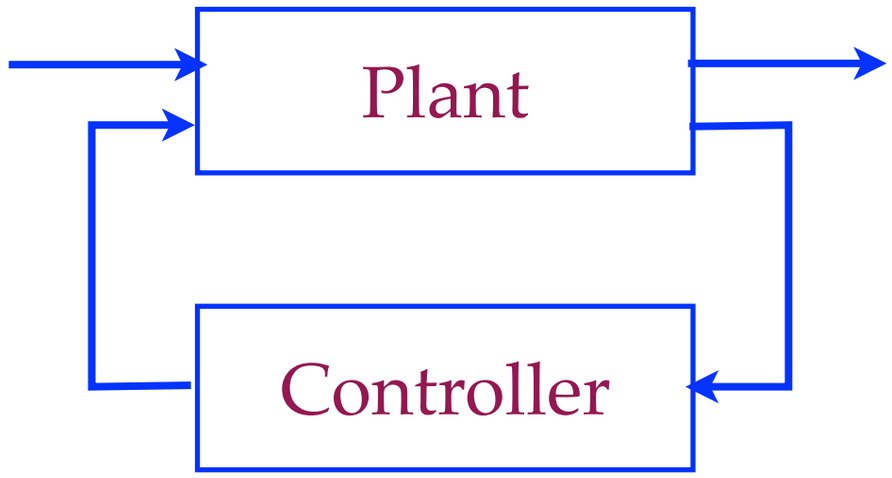
Model-based Design

- Model the plant



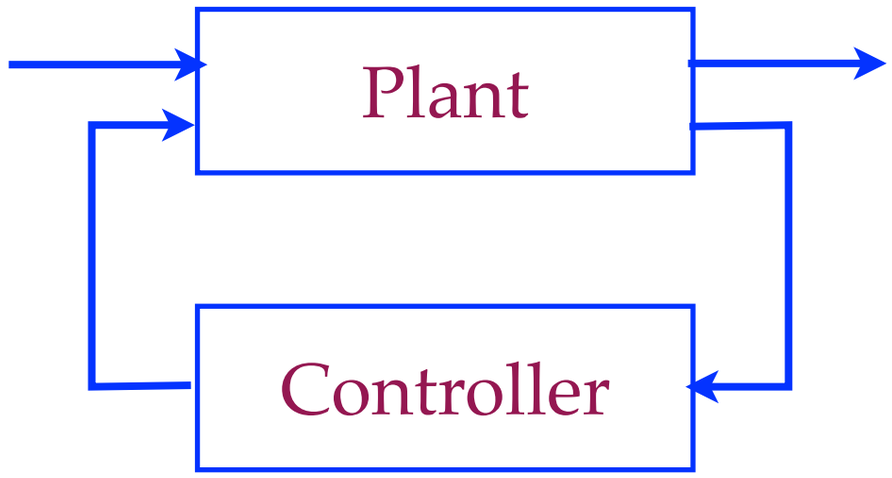
Model-based Design

- Model the plant
- Synthesize the controller



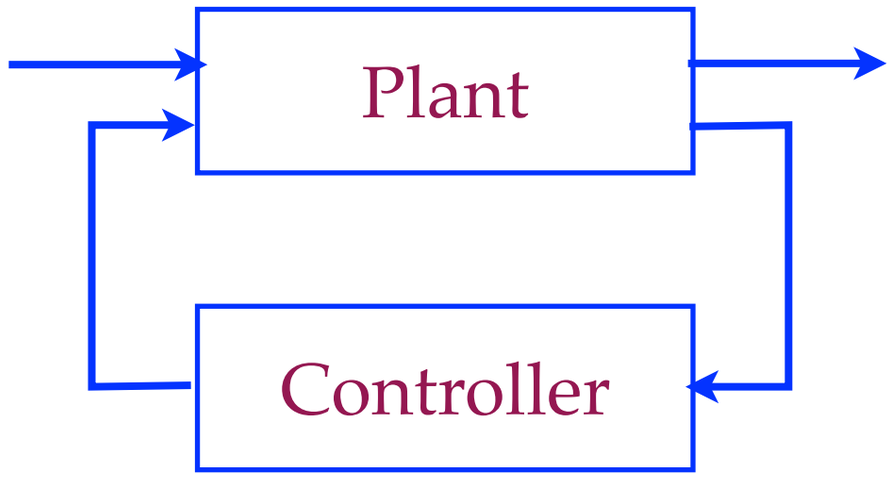
Model-based Design

- Model the plant
- Synthesize the controller
- Simulate / Verify



Model-based Design

- Model the plant
- Synthesize the controller
- Simulate / Verify
- Implement



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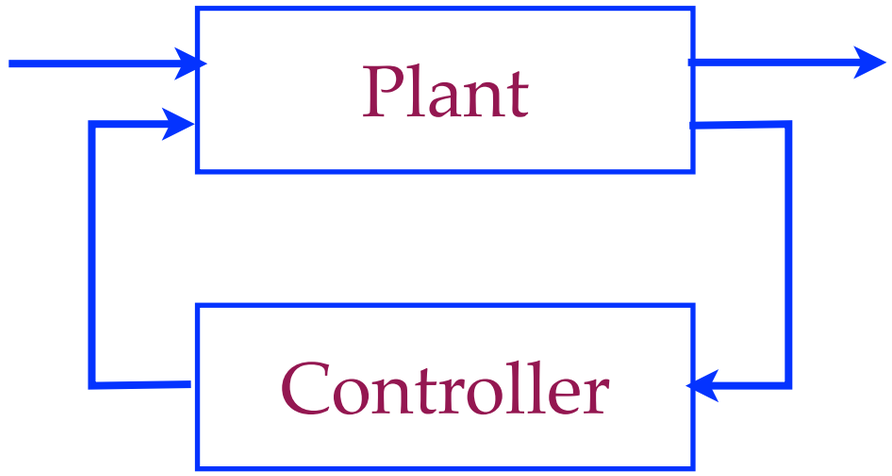
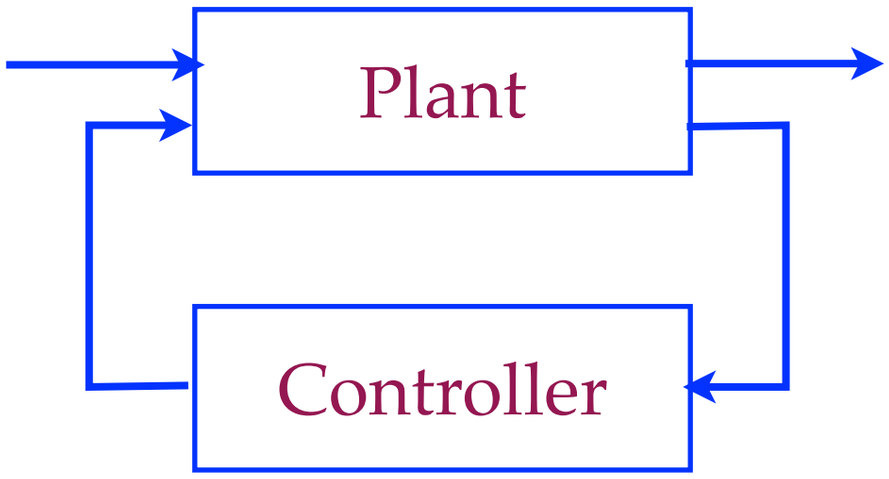
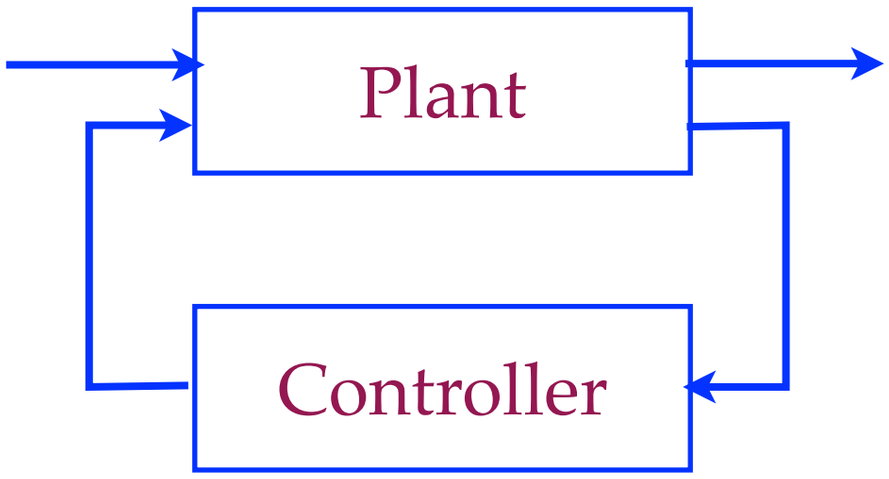


Elimination of errors early in the design, resulting in more robust control system, fewer iterations in the development cycle and reduced development time and cost.

Reliable design

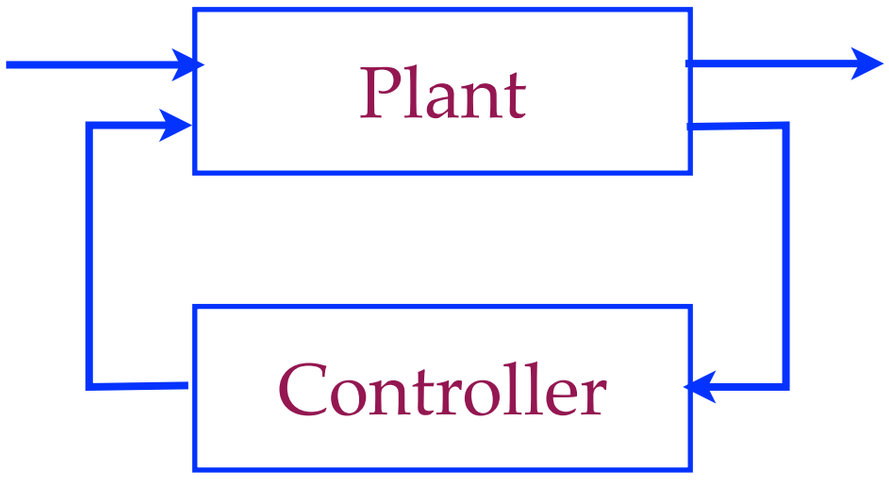


Reliable design



- Automated synthesis: Correct-by-construction

Reliable design



- Automated synthesis: Correct-by-construction
- Automated methods for detecting presence/absence of errors

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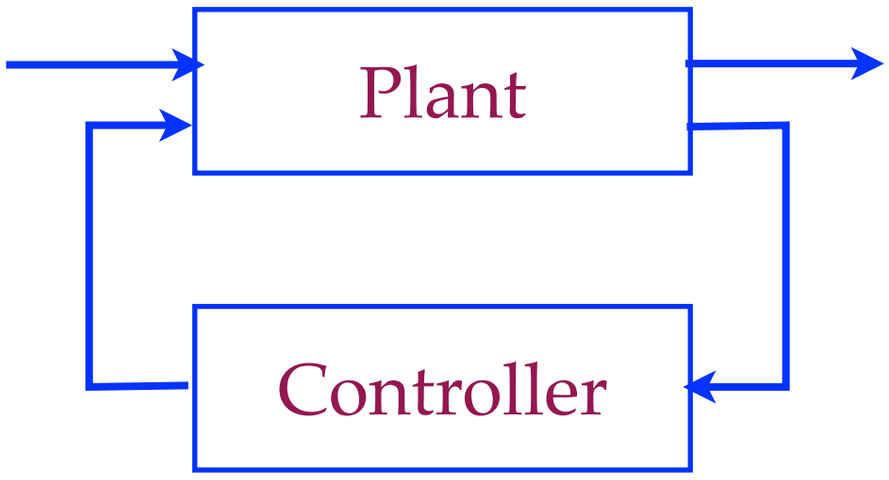

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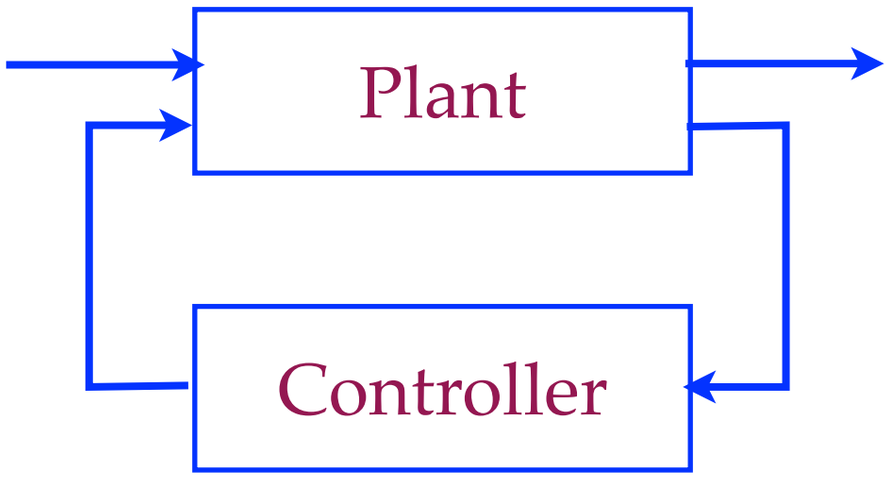
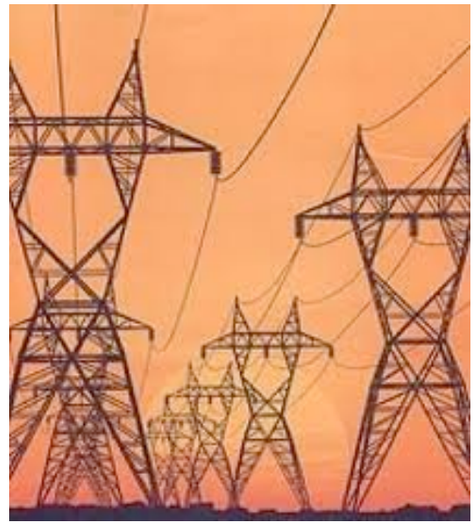


- Automated synthesis: Correct-by-construction
- Automated methods for detecting presence/absence of errors
  - Simulation/Testing
  - Verification

Verification of Cyber-Physical Systems

Hybrid Systems

Systems consisting of mixed discrete-continuous behaviors



Highlights

- Modeling and specification
- Overview of safety verification
- Overview of stability verification
- Complexity and computability
- Approximation techniques
- Tools

Models

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- Hybrid Automata [Henzinger et al.]

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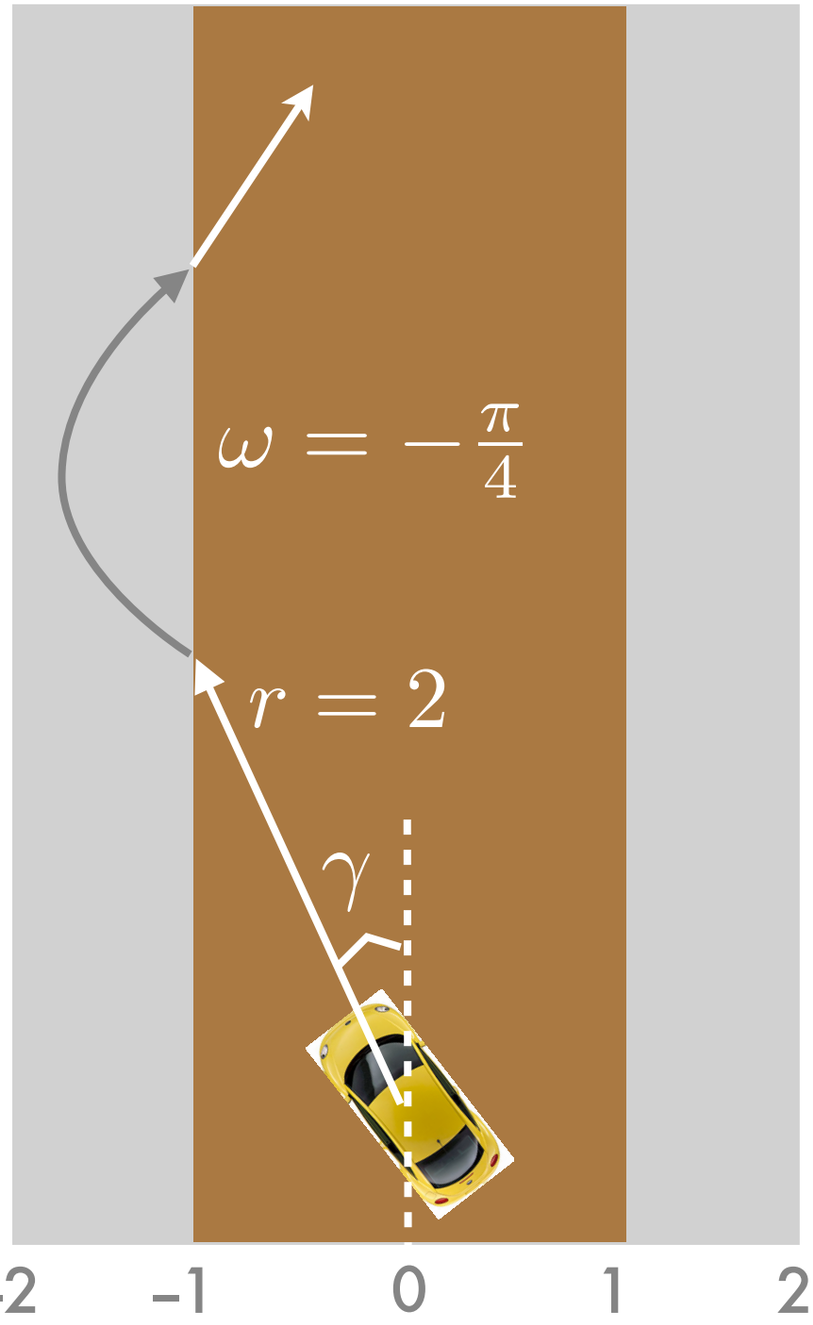
Models

- Hybrid Automata [Henzinger et al.]
  - Continuous dynamics: Differential equations
  - Discrete dynamics: Finite state automata
- Extensions of process algebra and petri-nets, differential dynamic logic

Autonomous Car Controller



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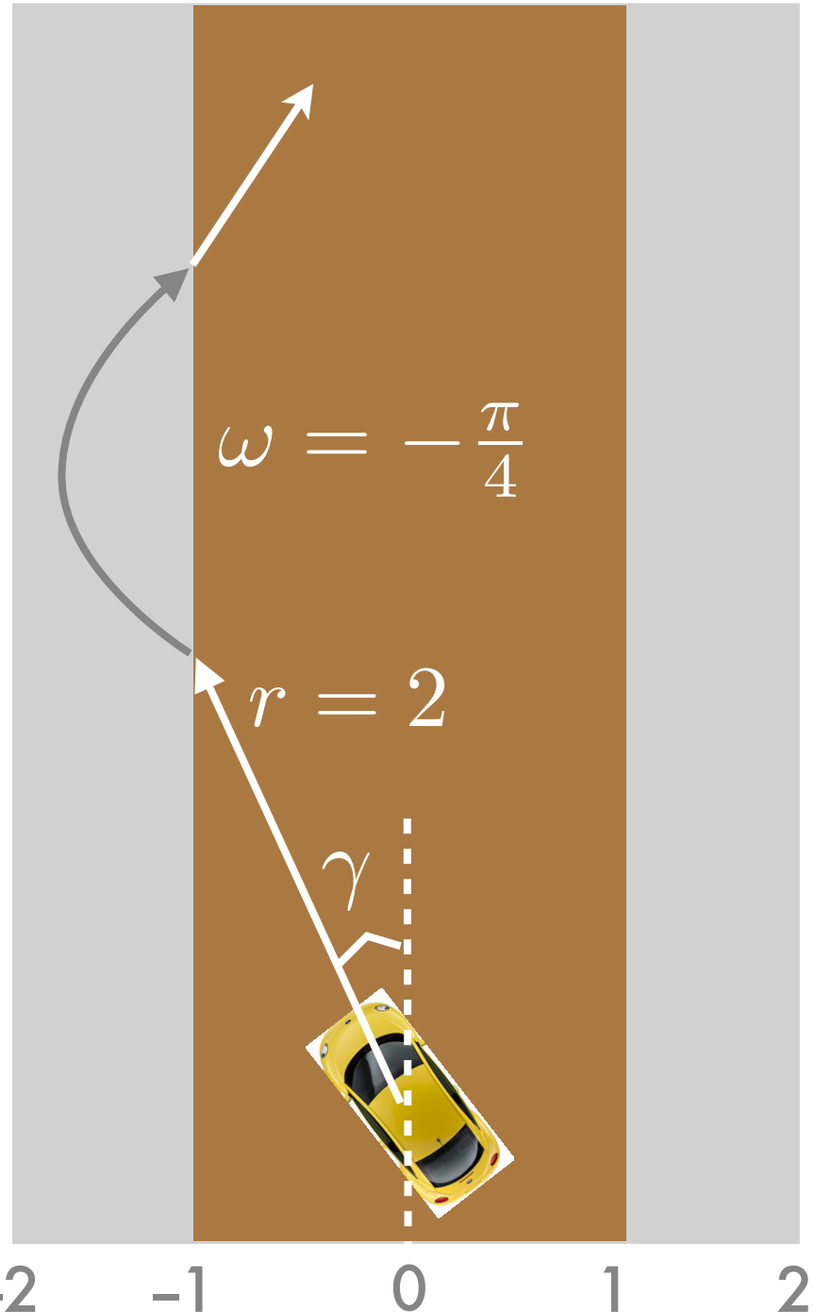

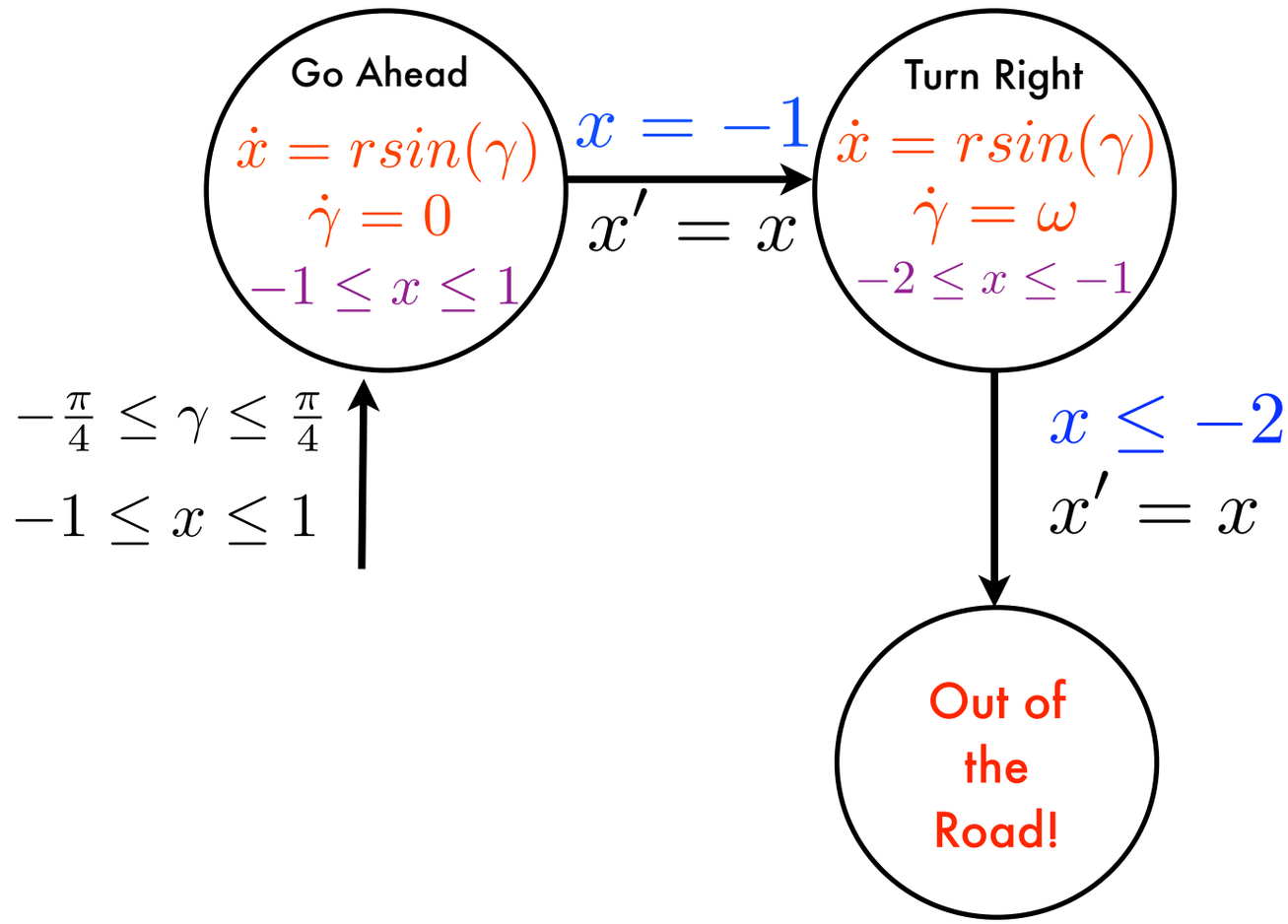

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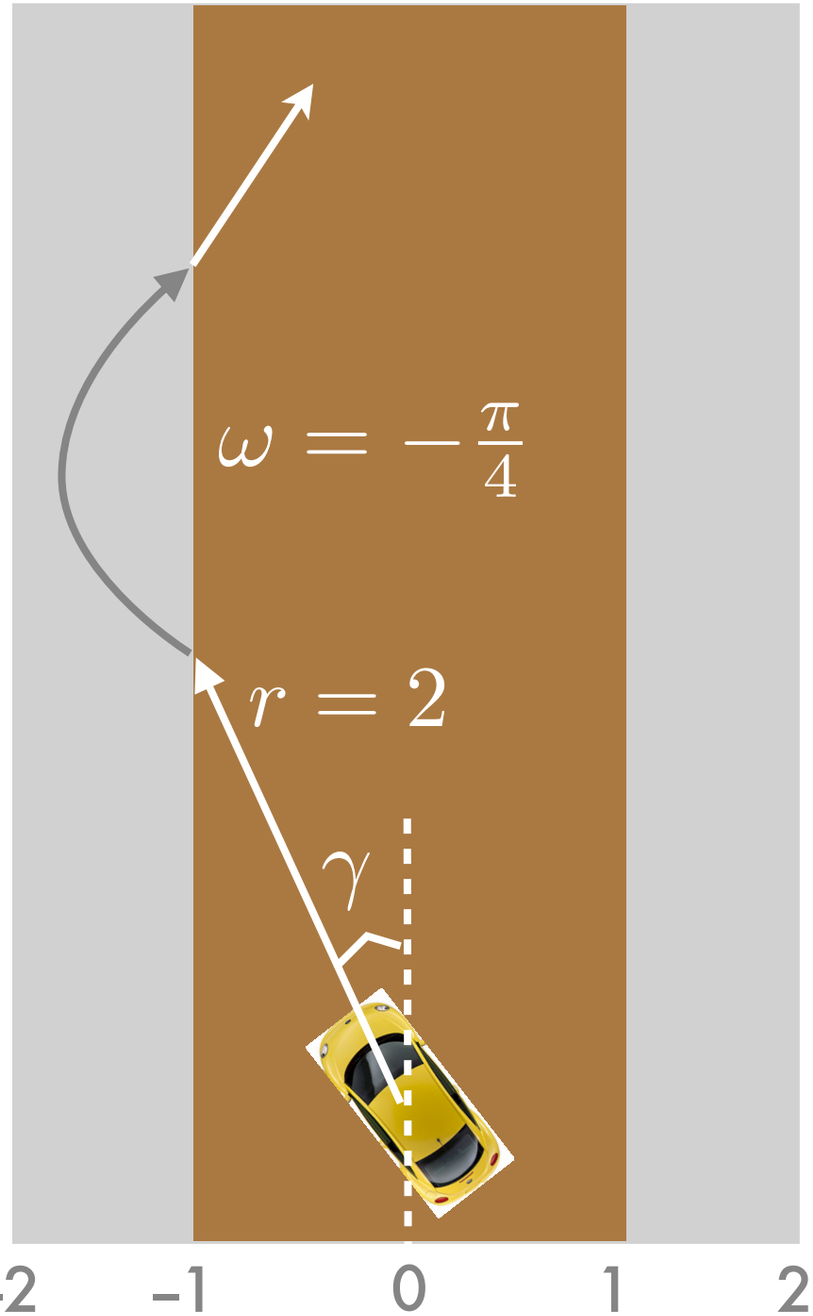

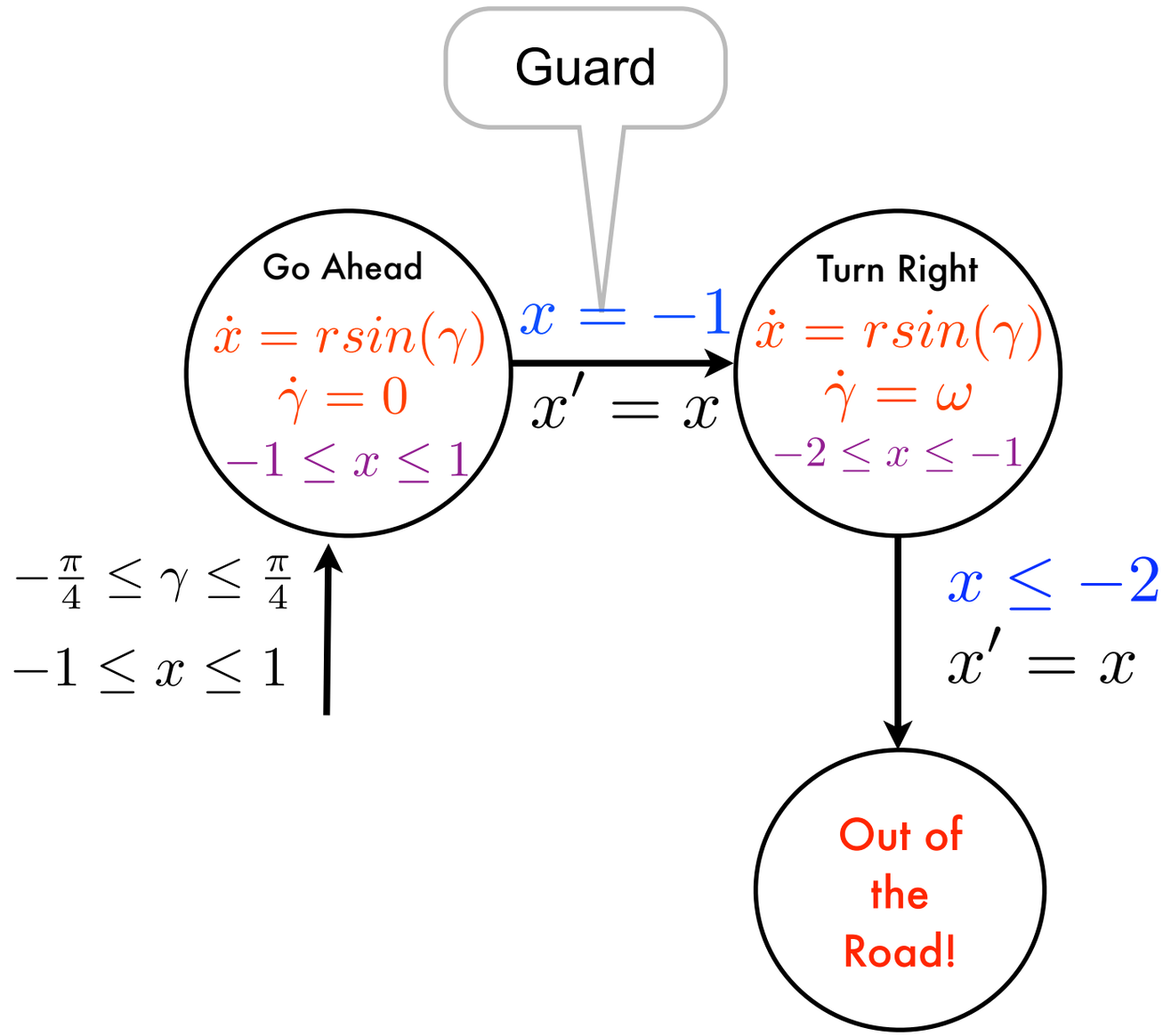

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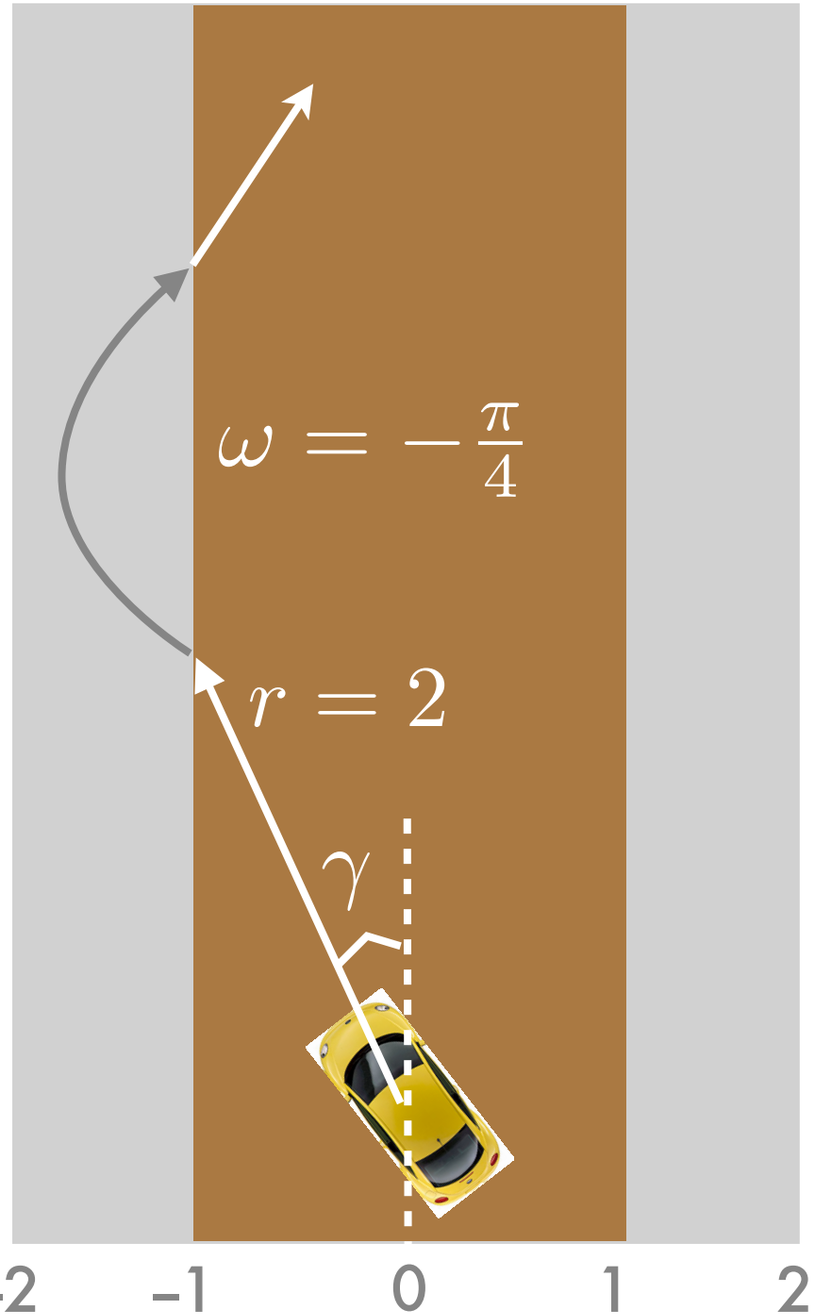

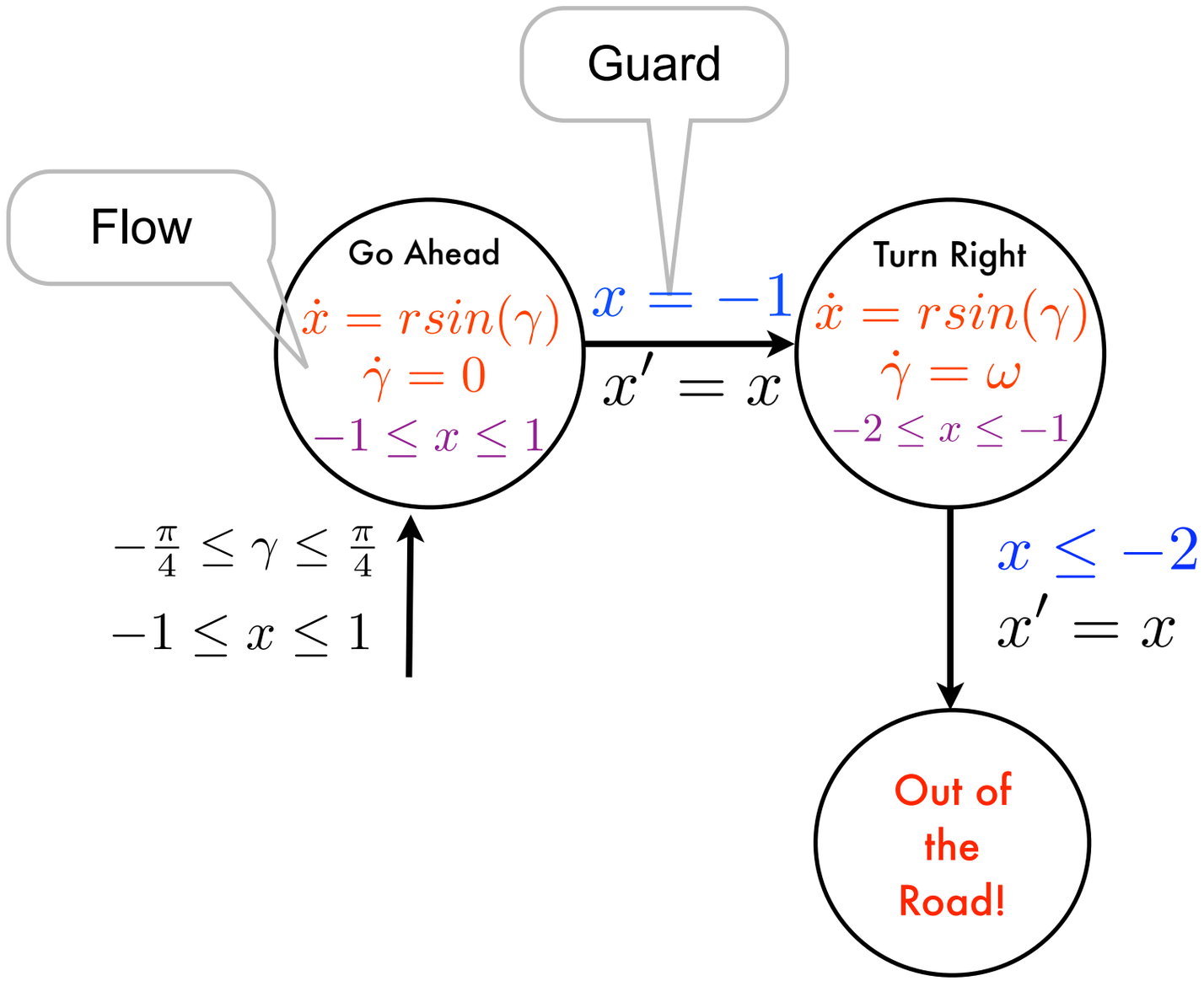

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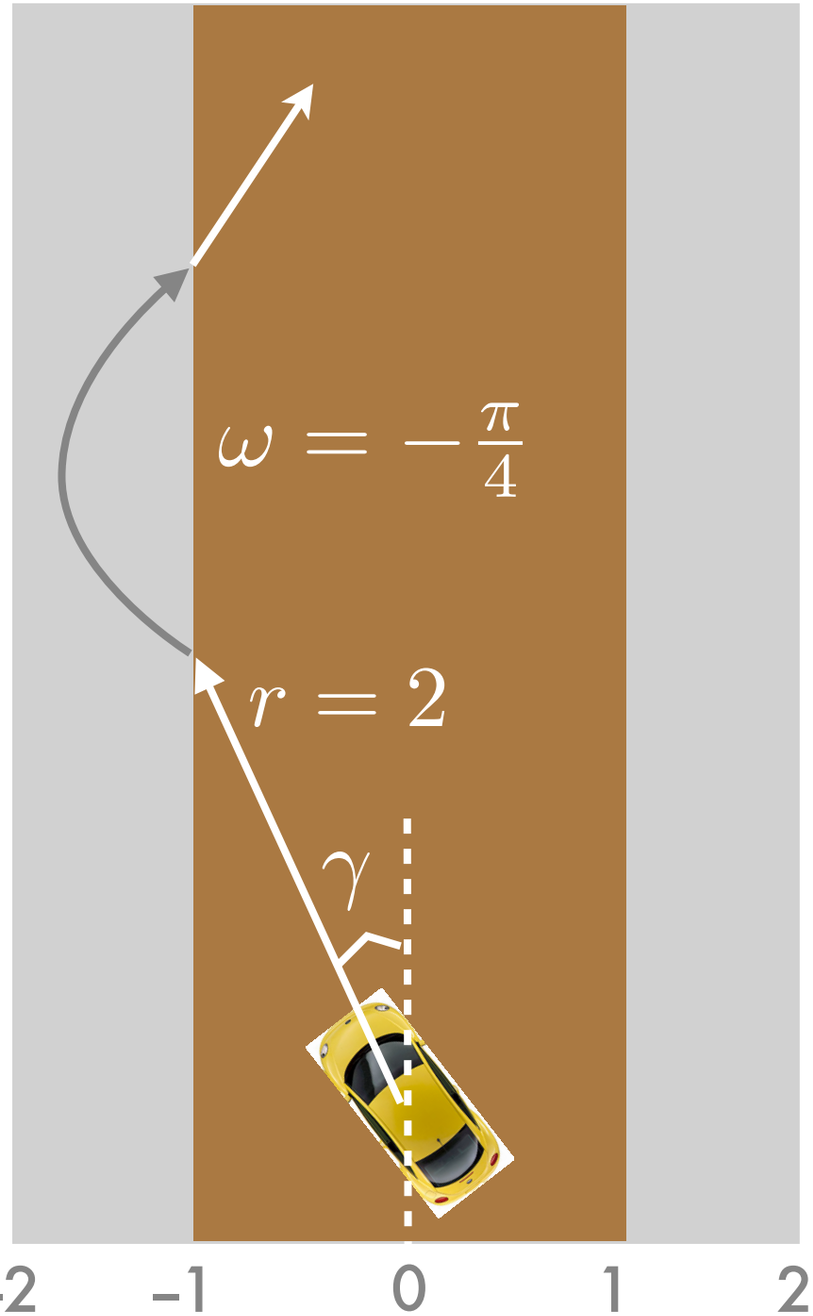

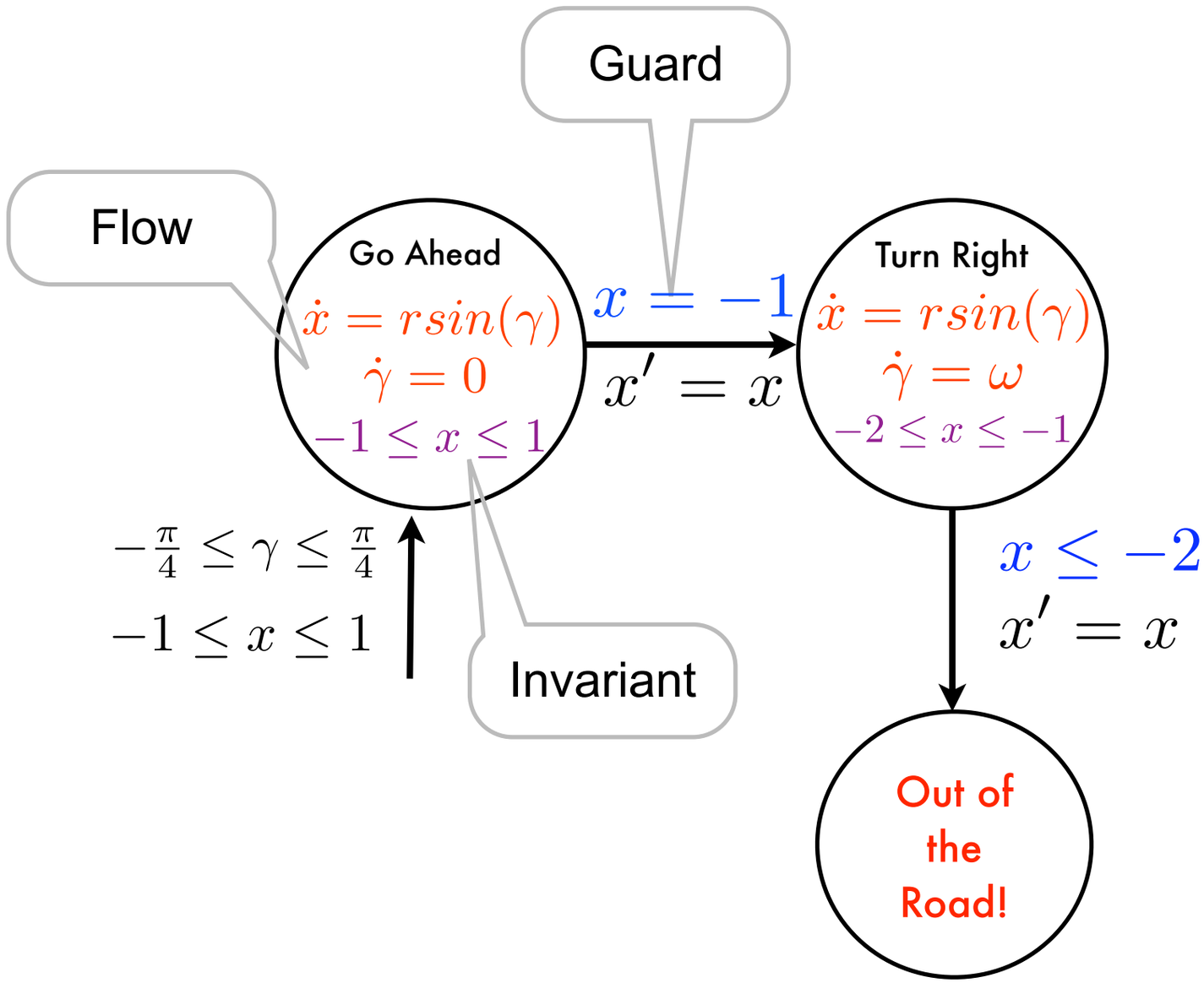

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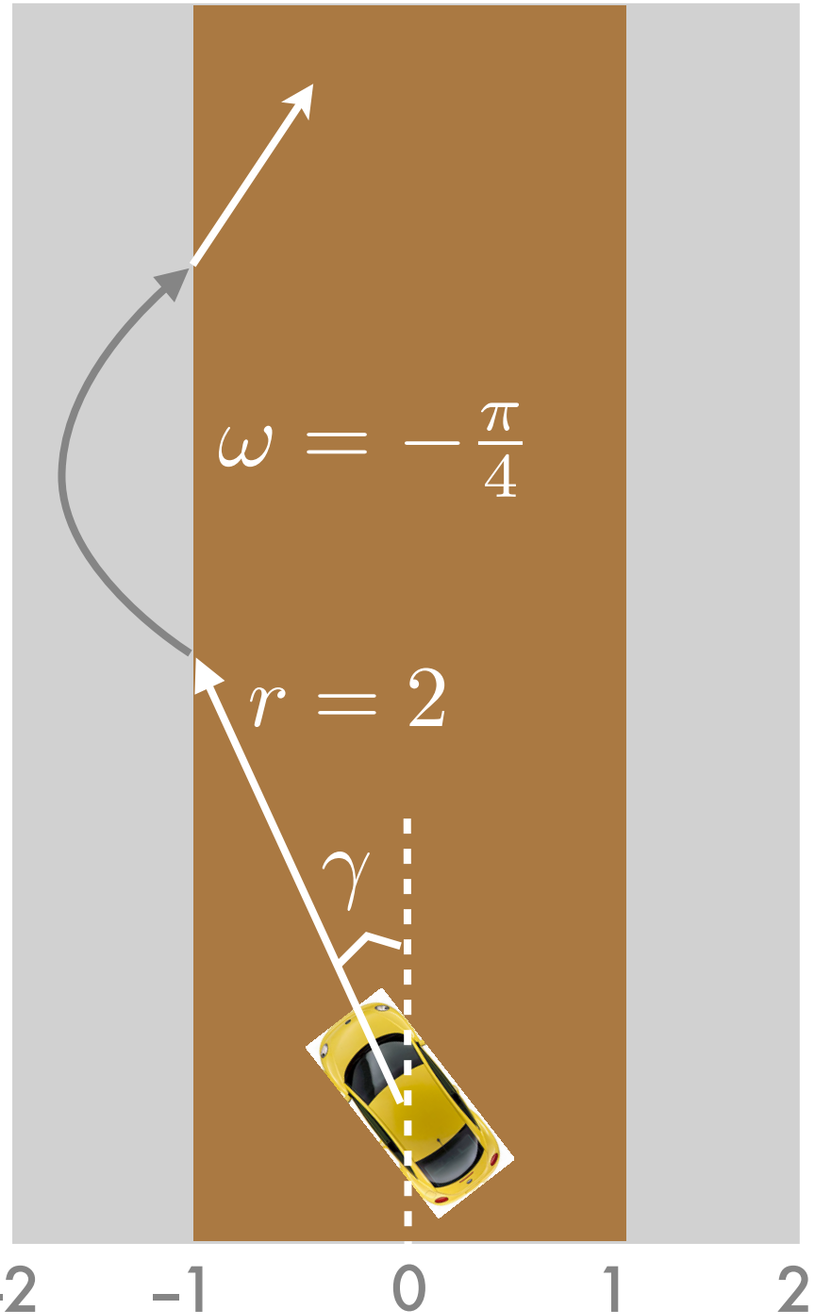

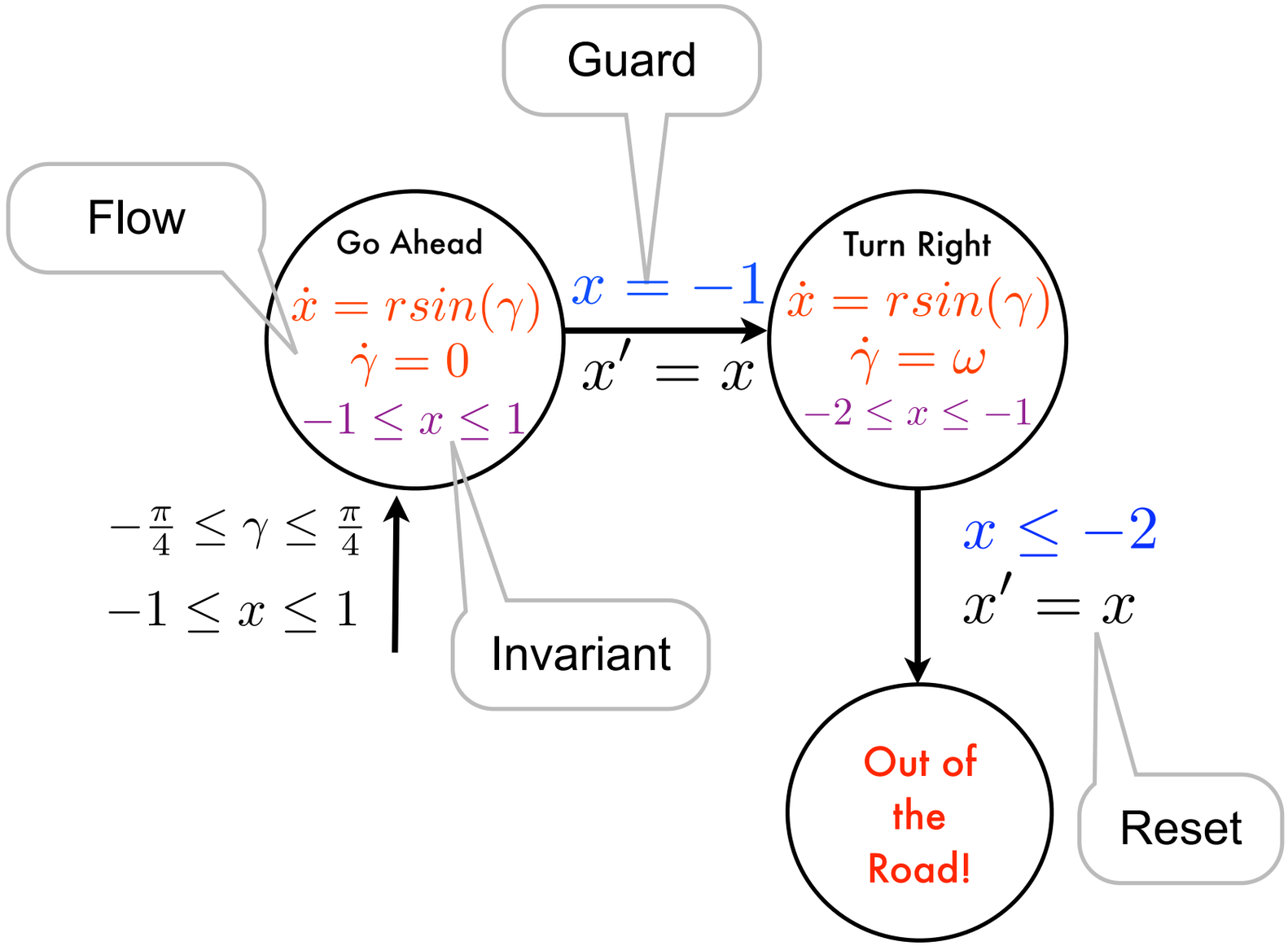

Properties

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Stability

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Safety

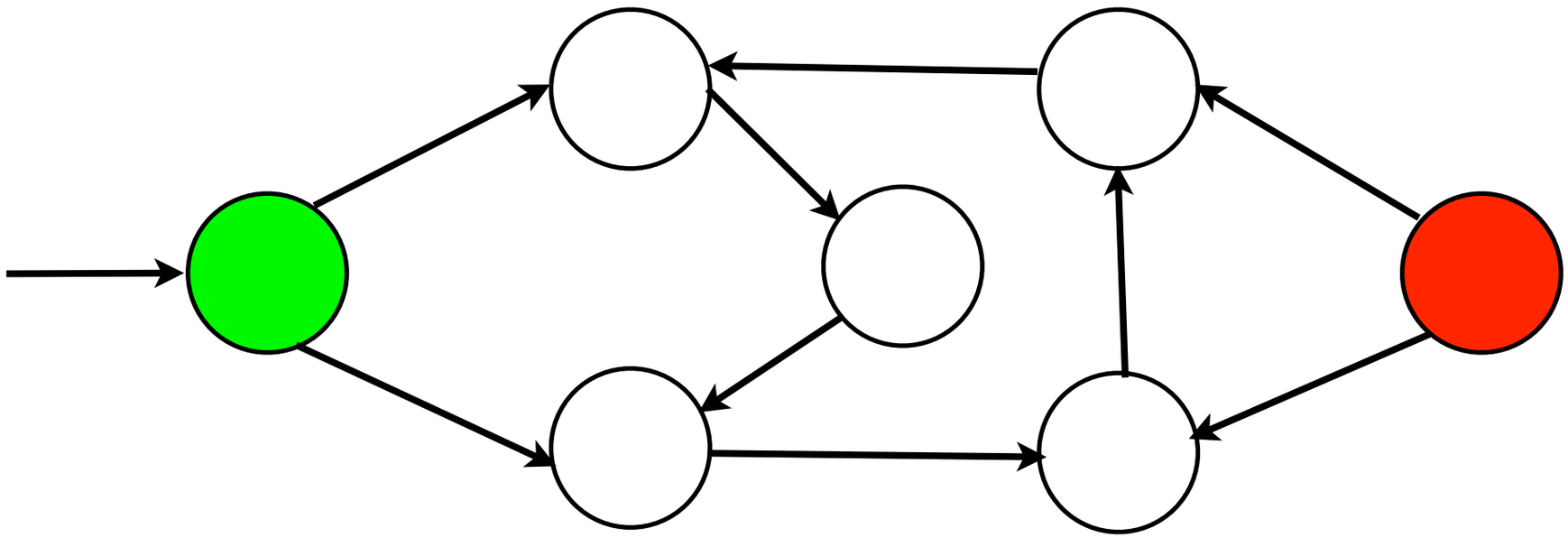
- Every execution of the system is error free
- The car does not go out of the road

Stability

- Small perturbations in the initial state or input lead to only small perturbations in the eventual behavior of the system
- Small perturbations in the initial orientation of the car will still keep it inside the road

Overview of safety verification

Finite-state systems & model-checking



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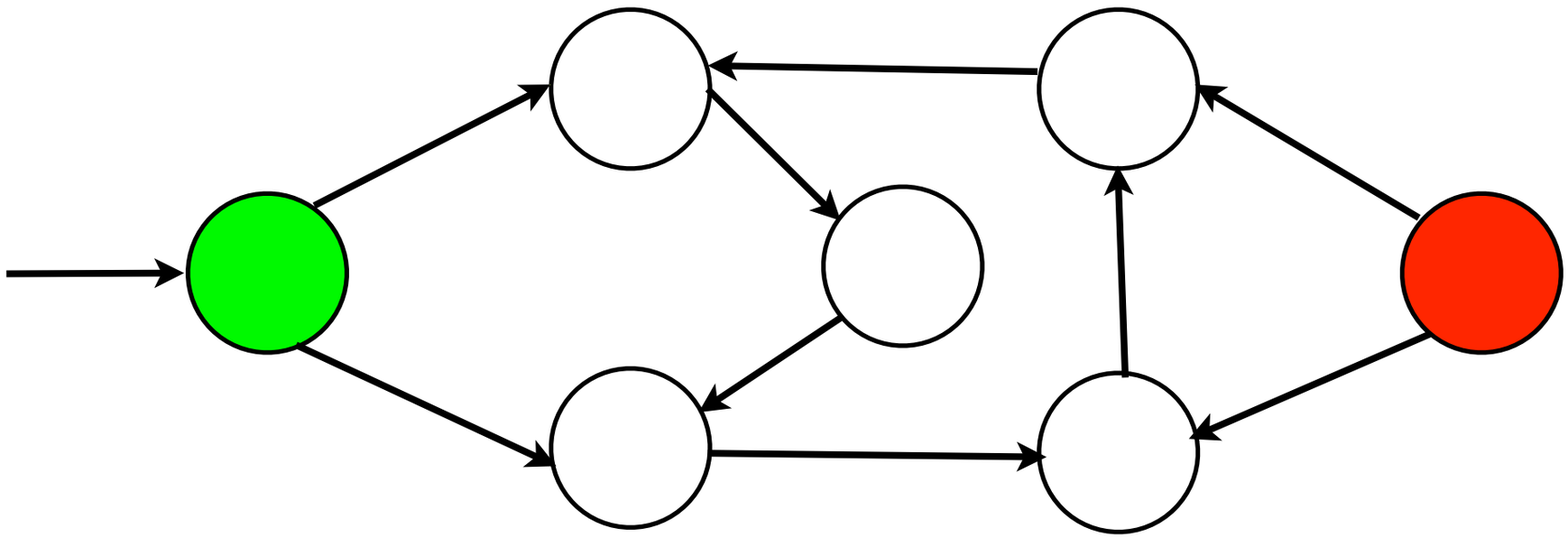
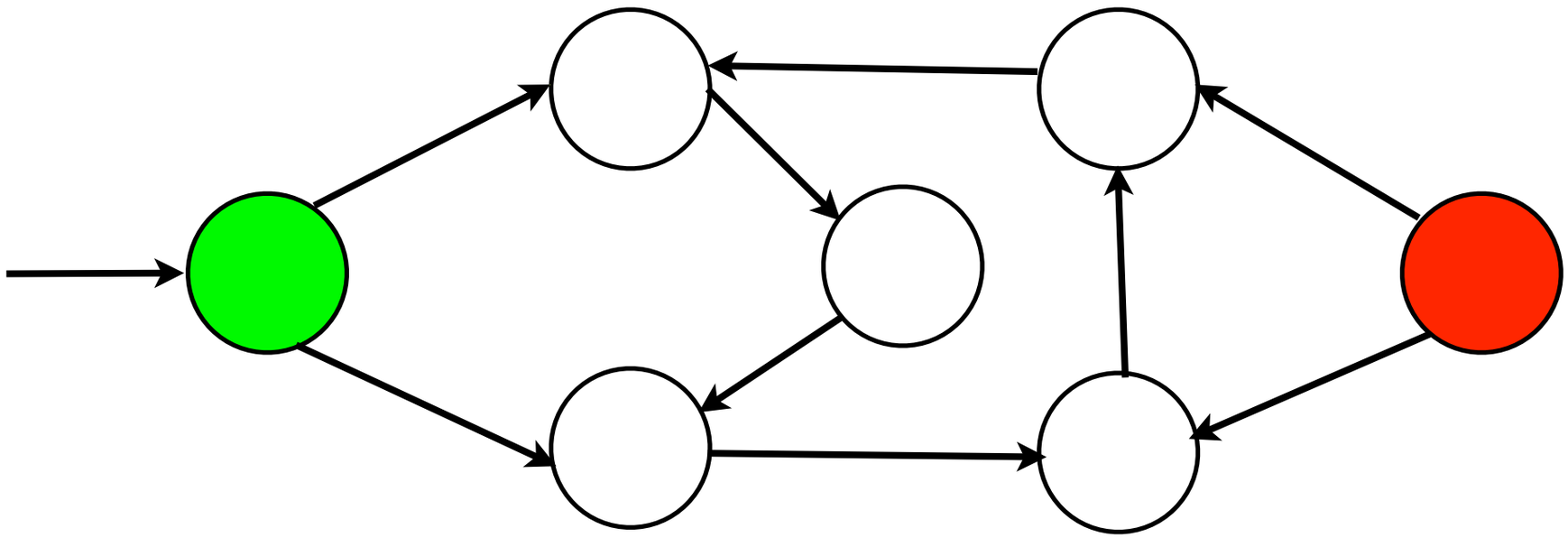


- Exhaustive state space exploration

Finite-state systems & model-checking



- Exhaustive state space exploration
- Compute one step-successors

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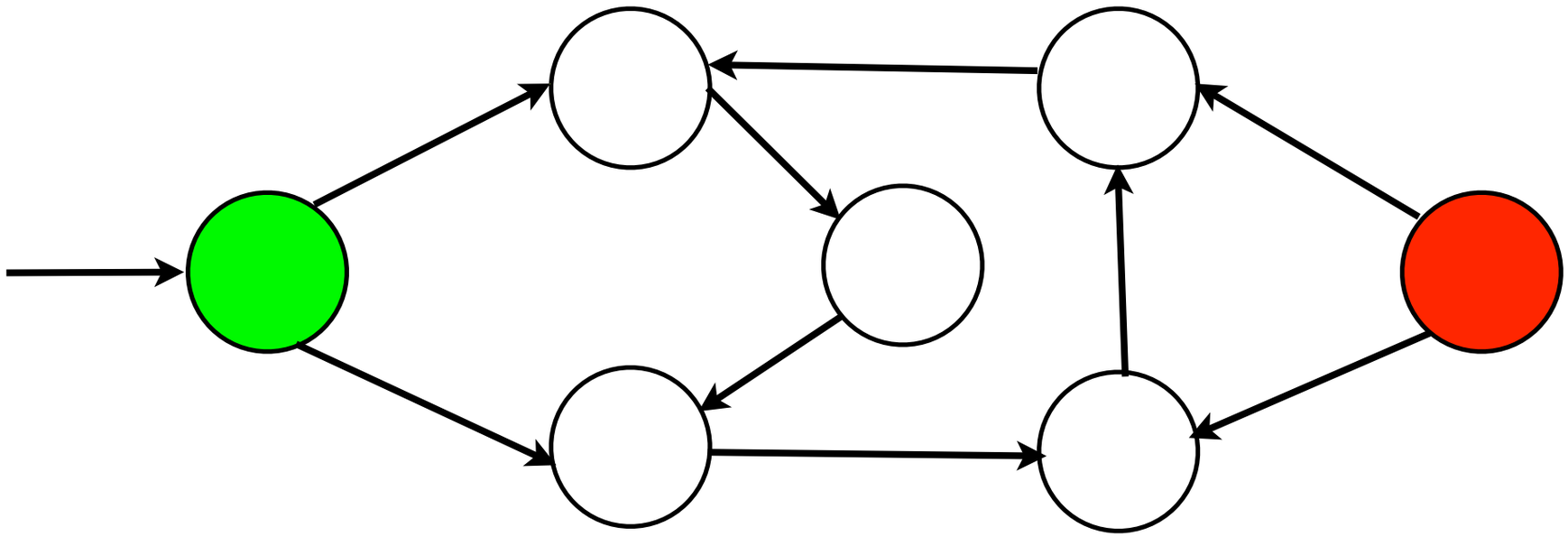

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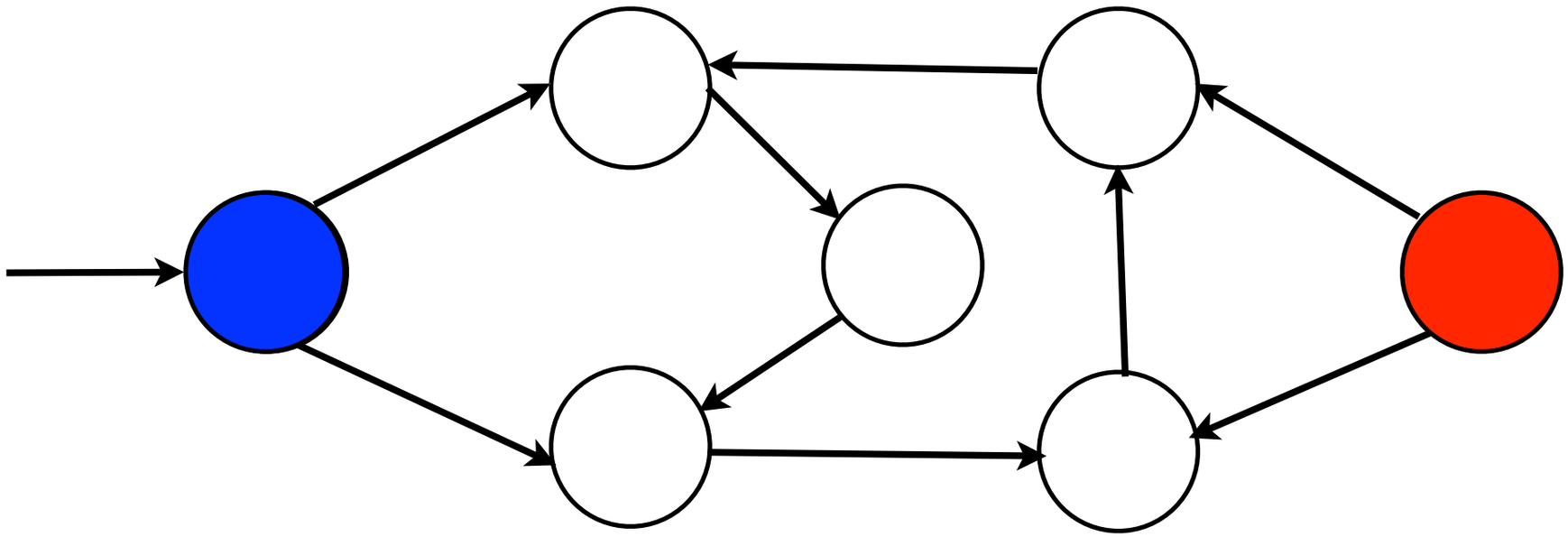

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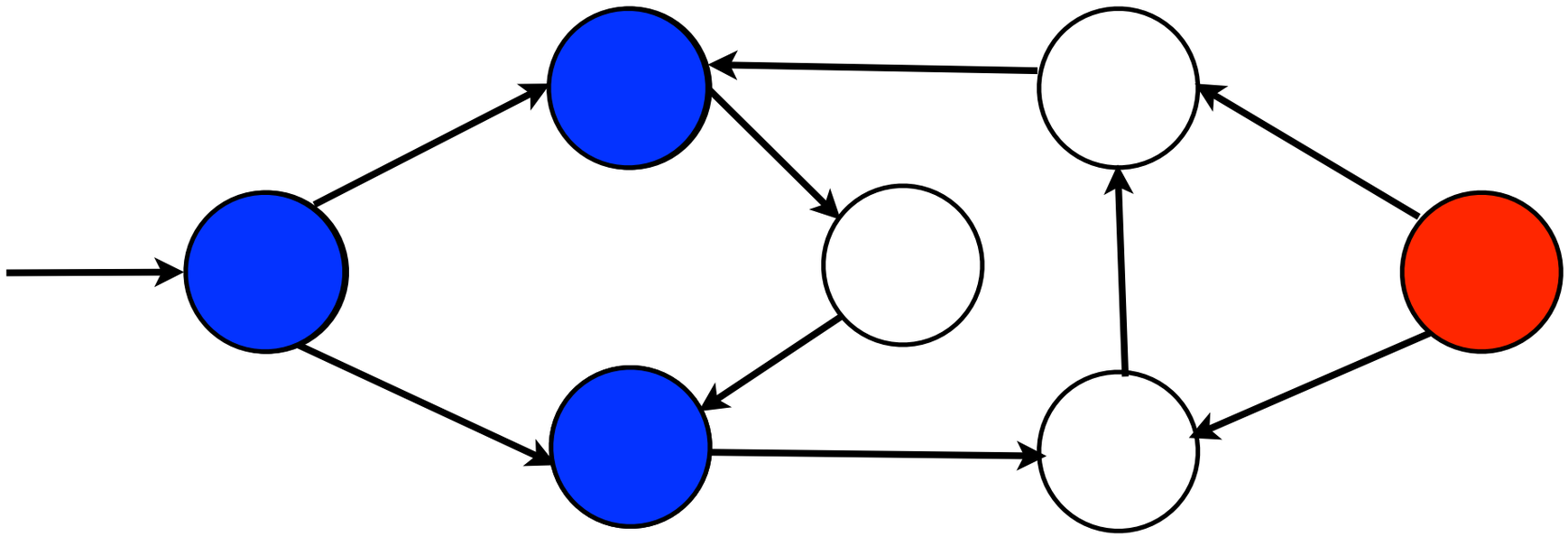

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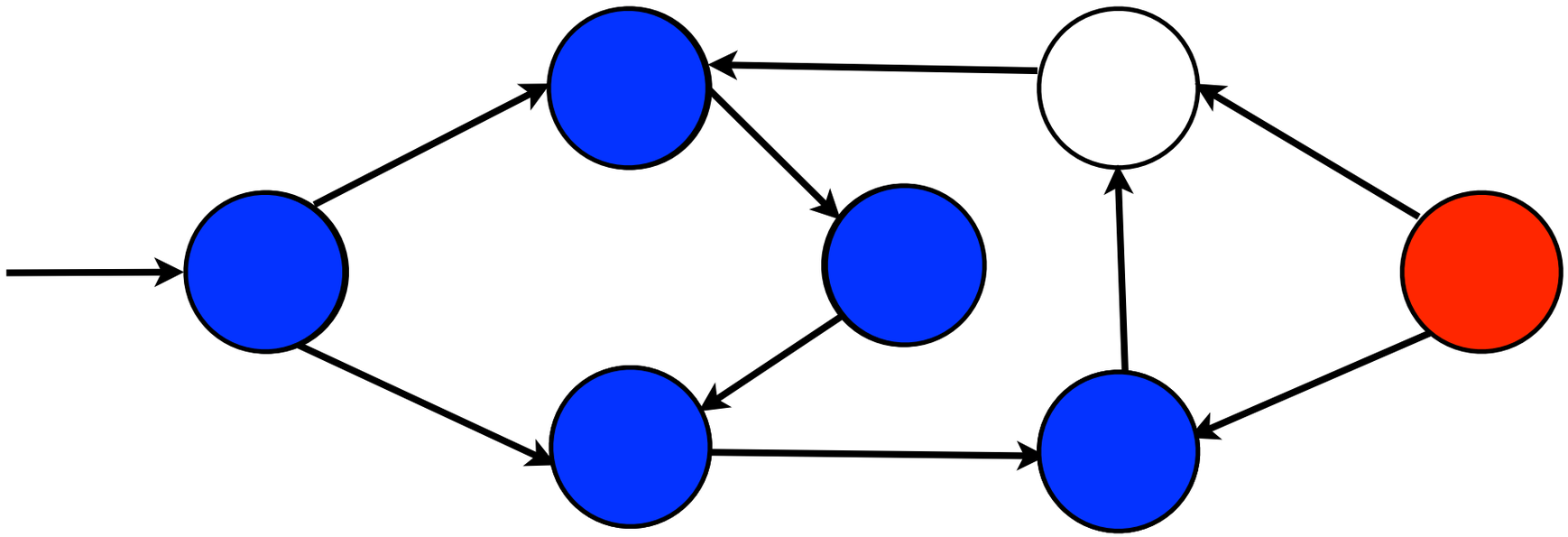

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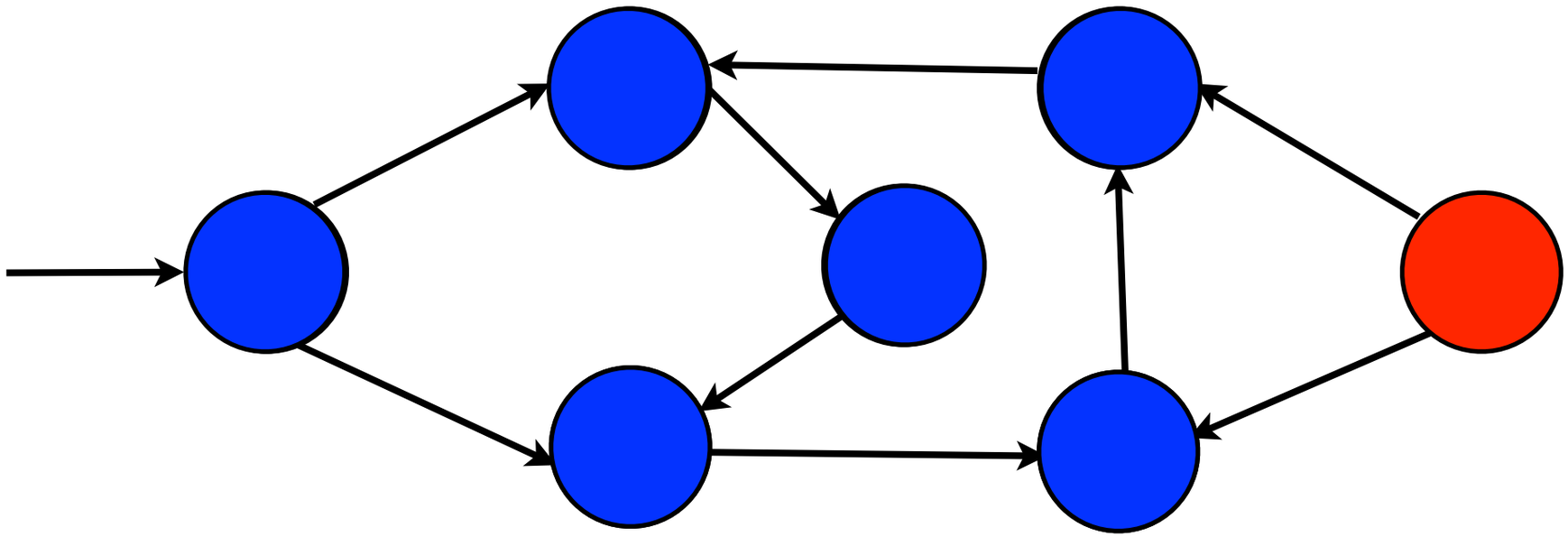

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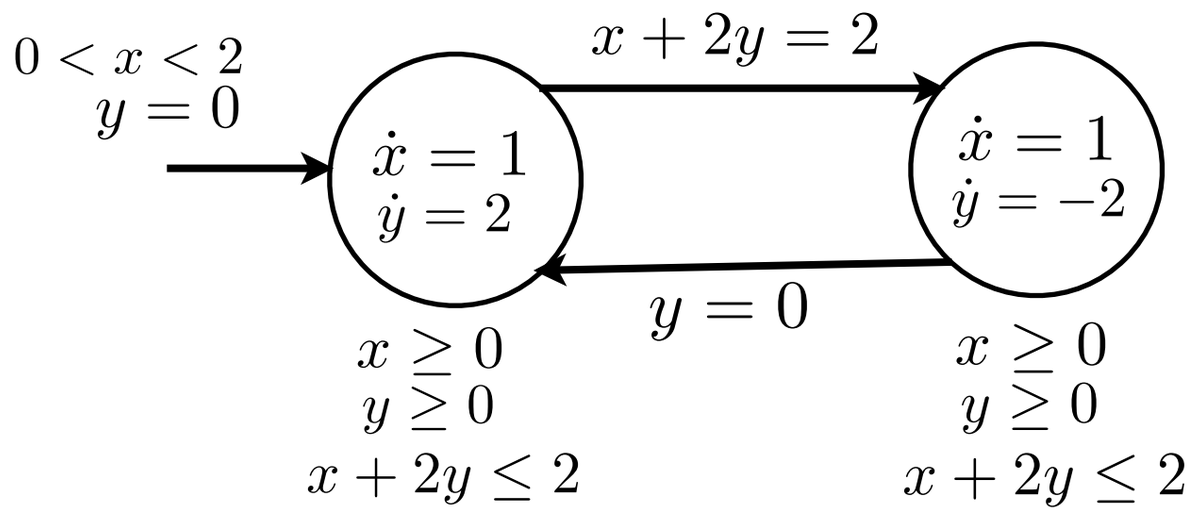
A simple class of hybrid systems



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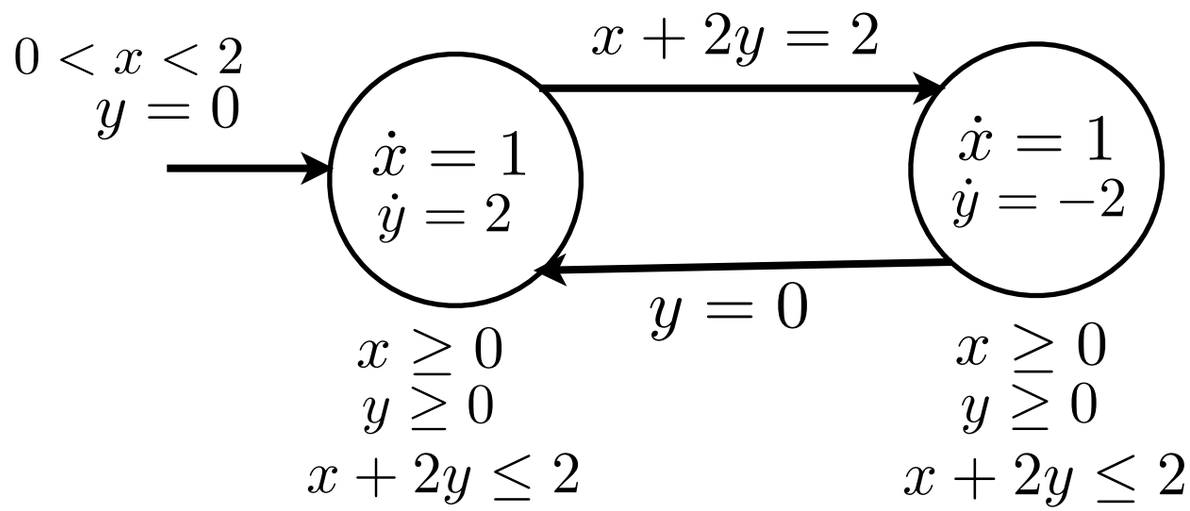

- Continuous dynamics - constant derivative
- Invariants and guards - linear
- Resets - identity

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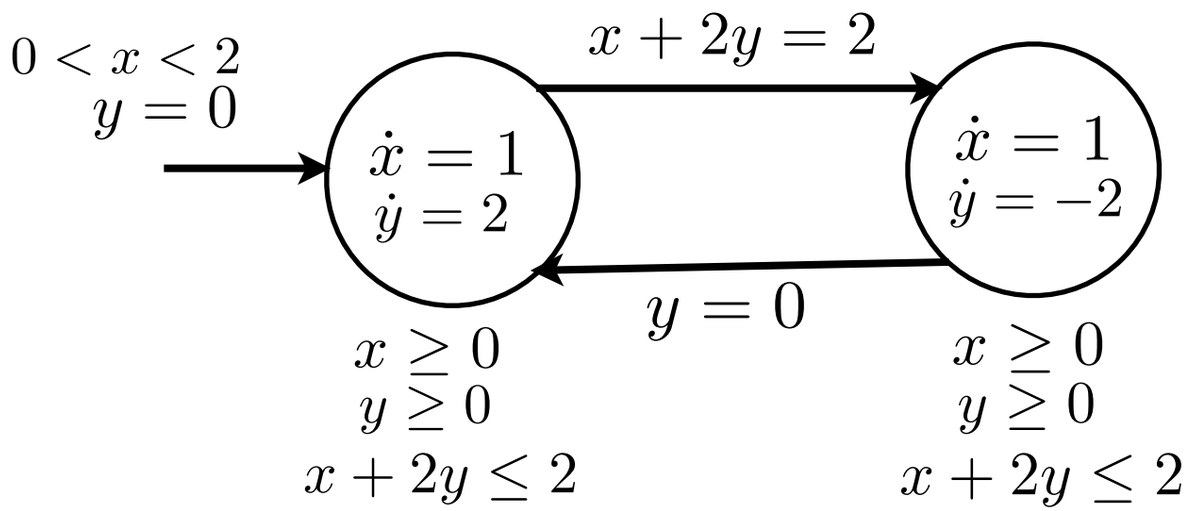

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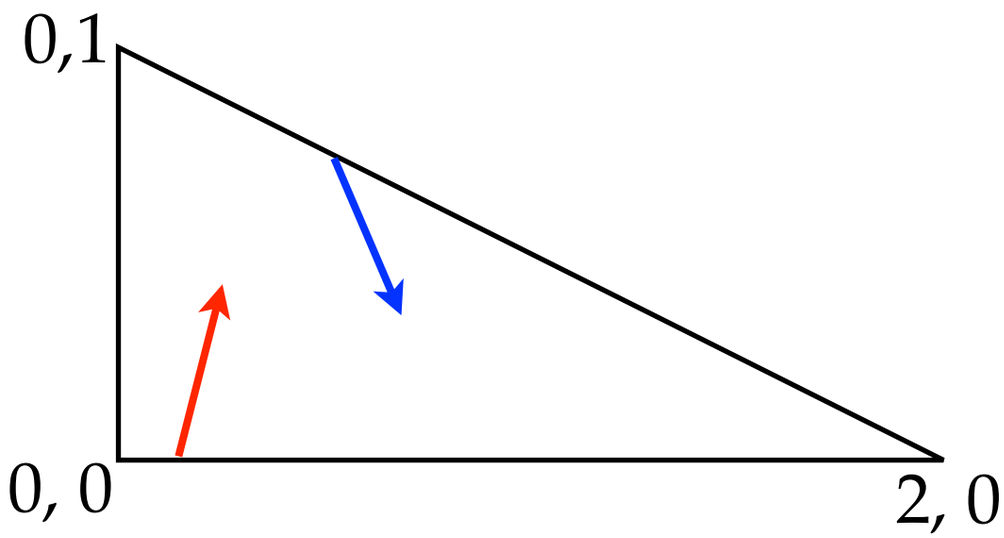

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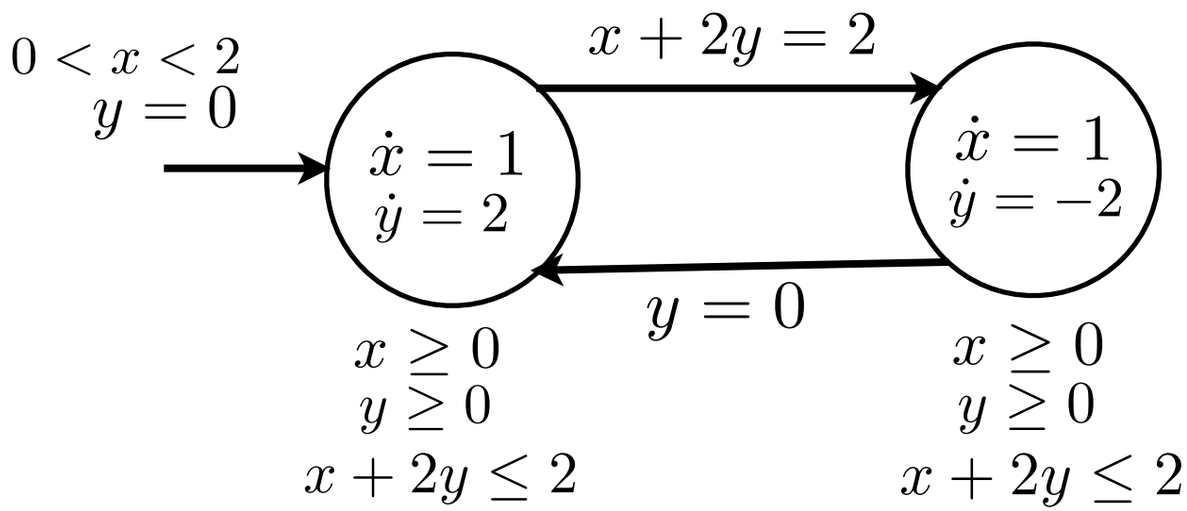

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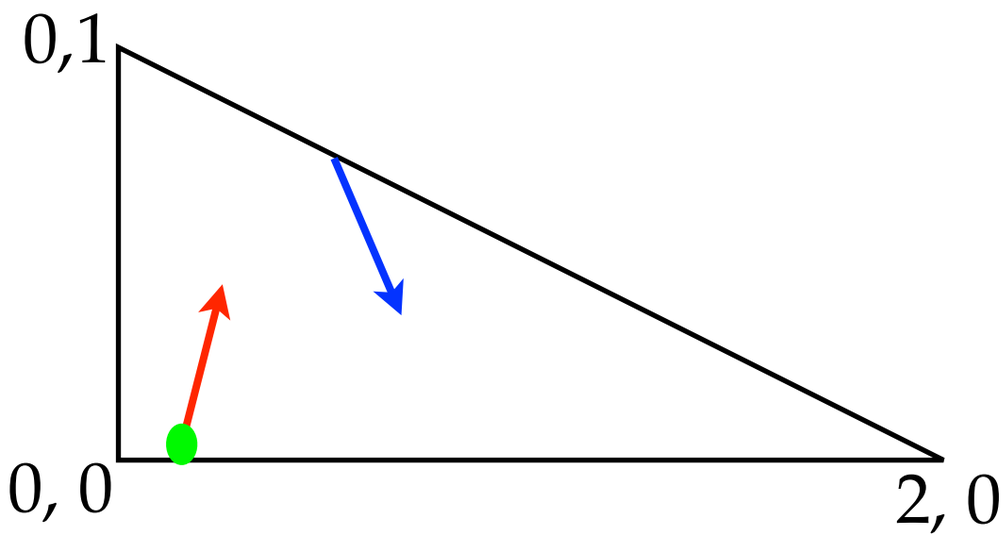

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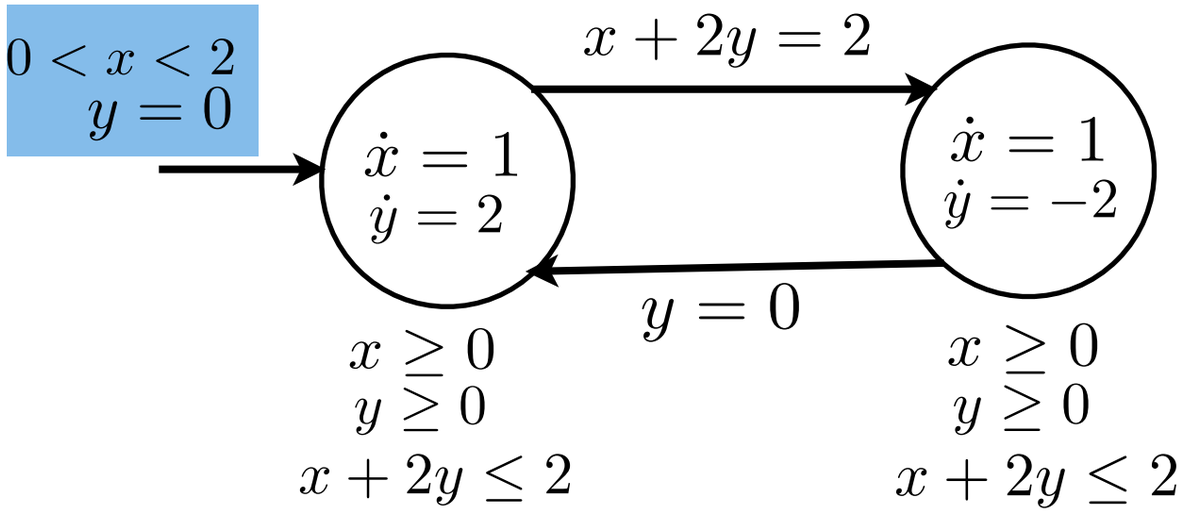

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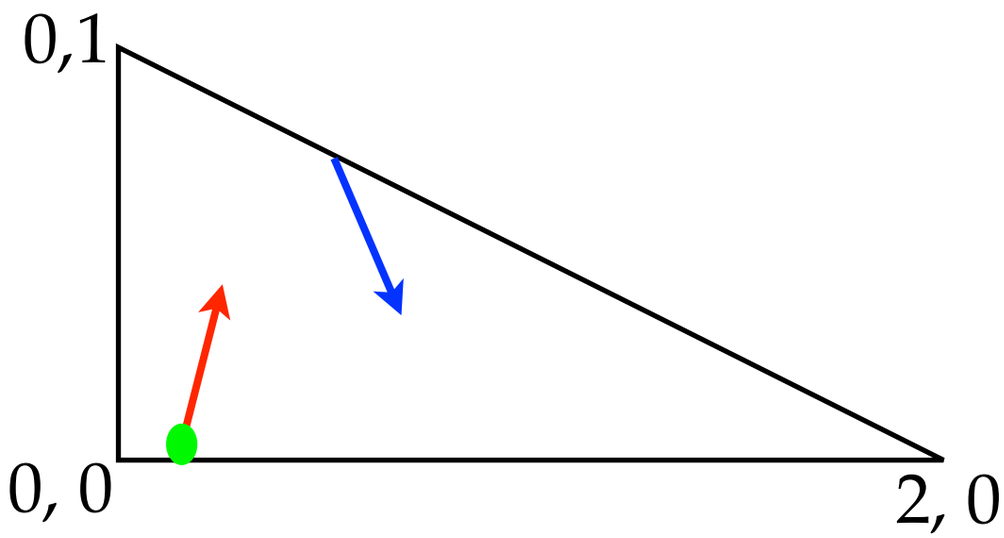

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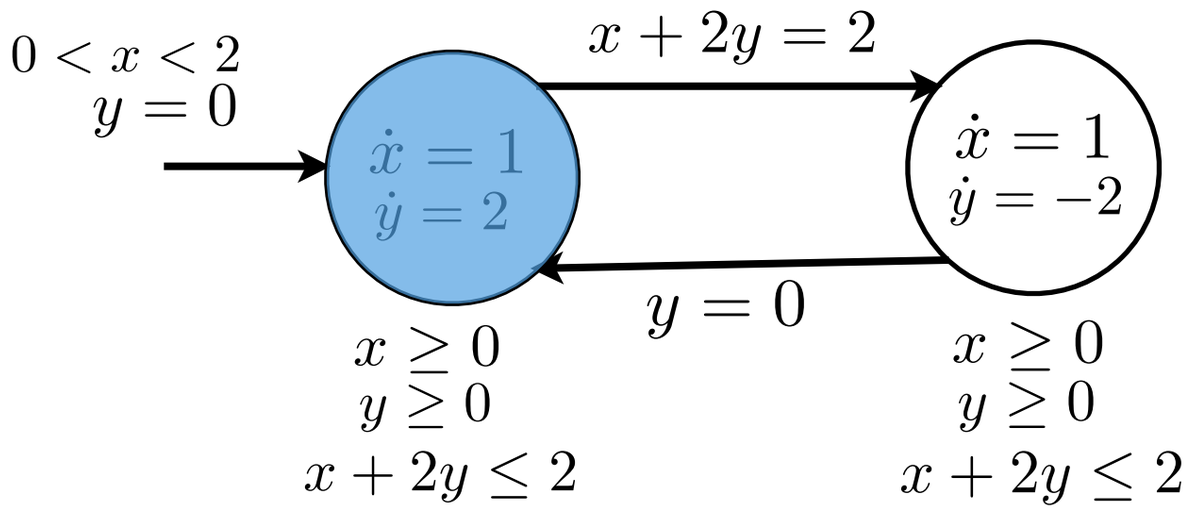

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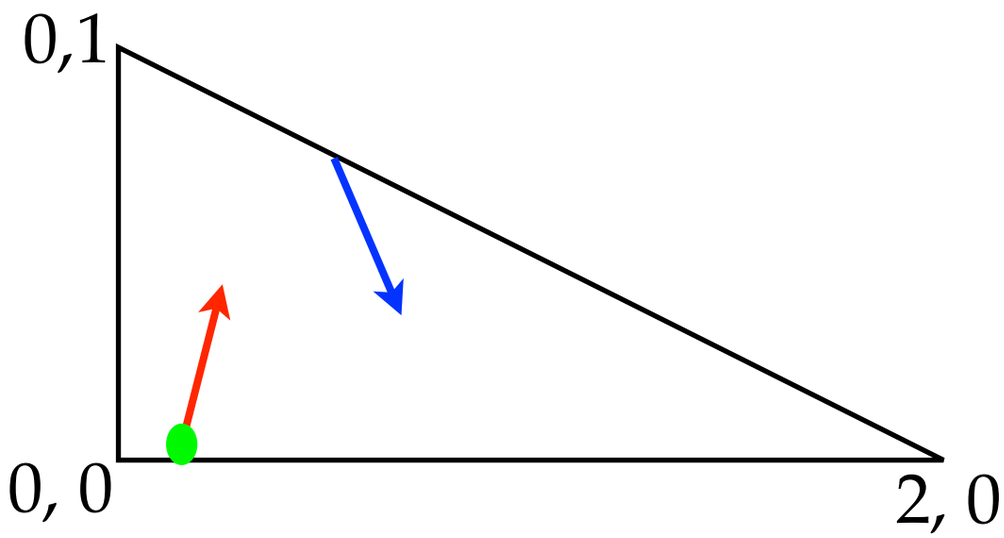

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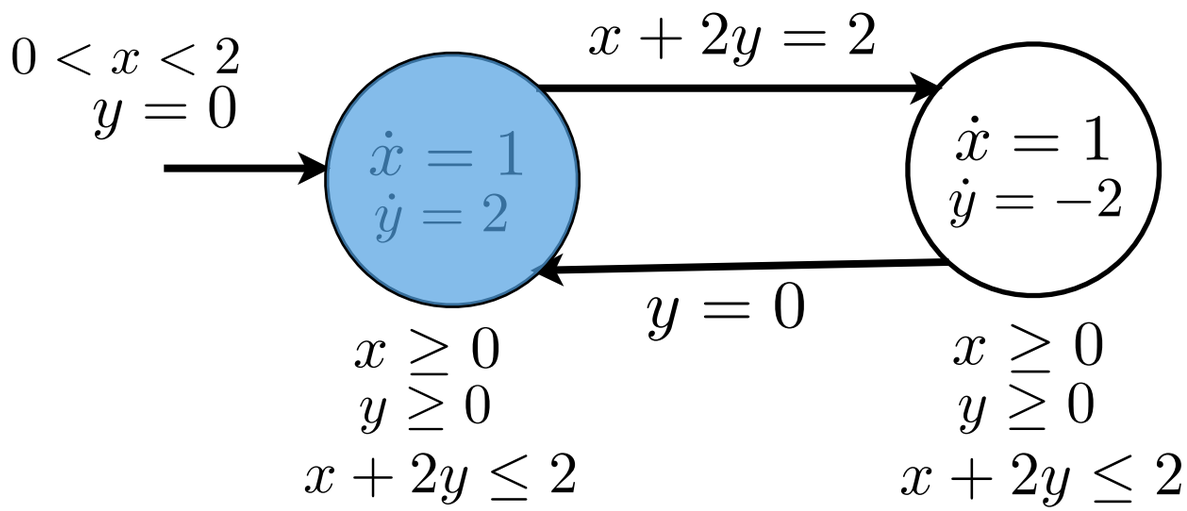

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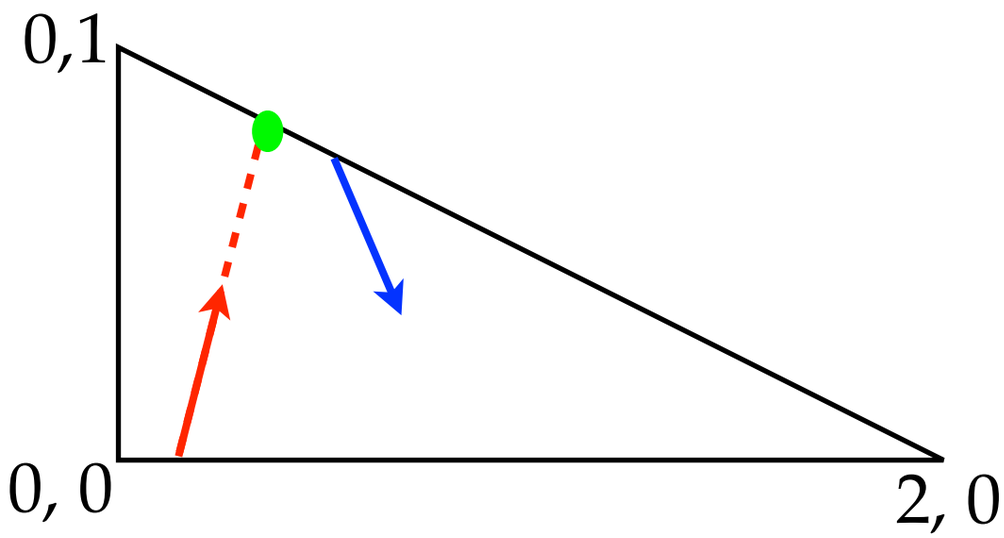

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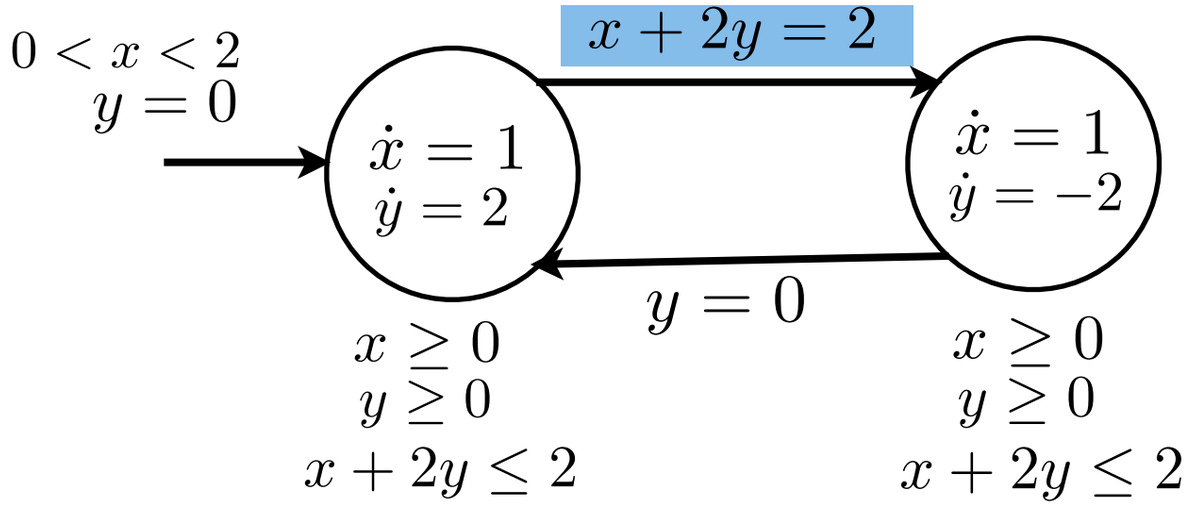

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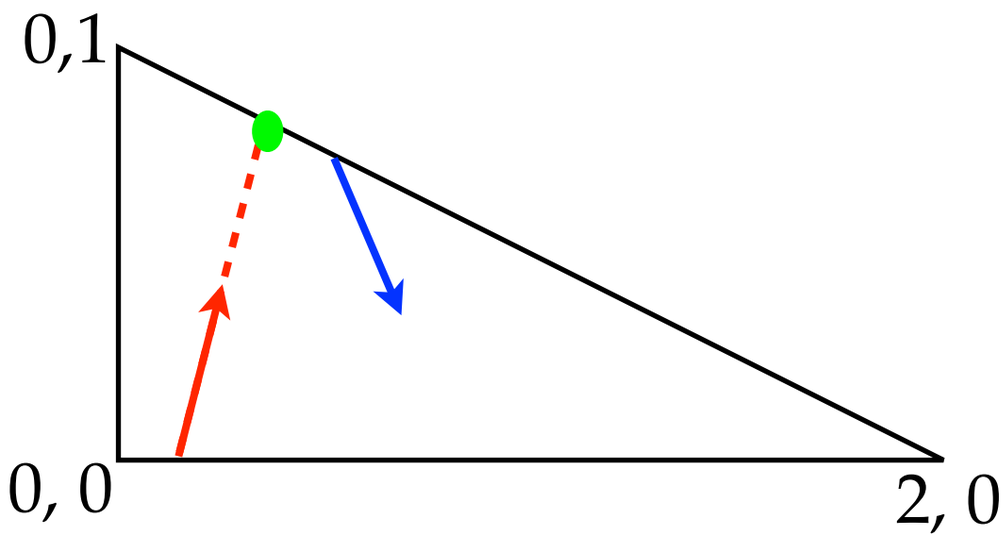

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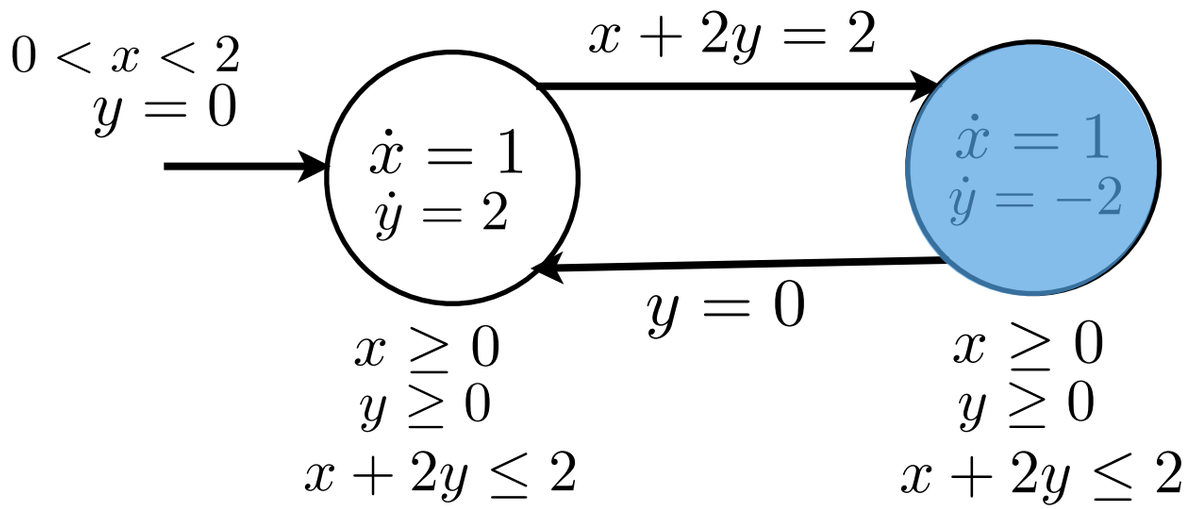

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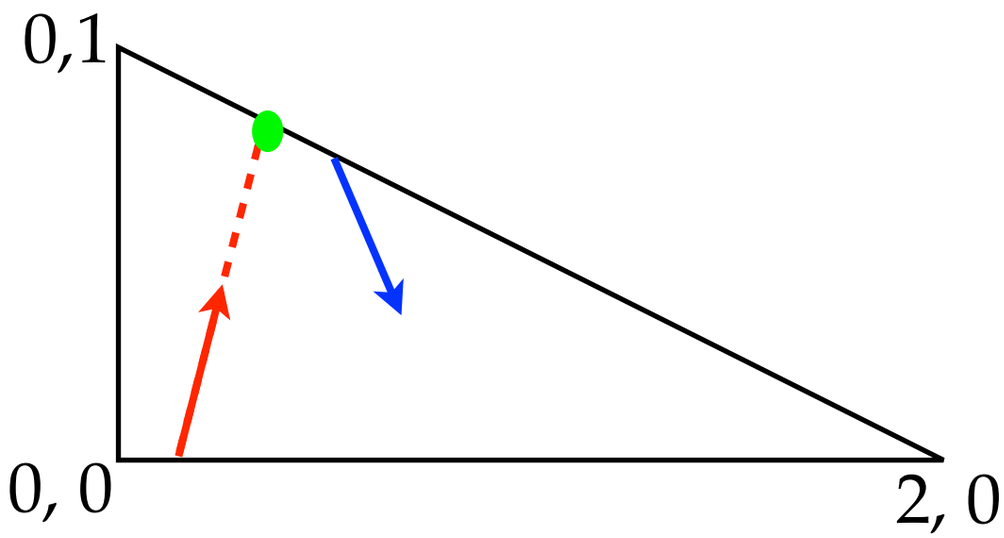

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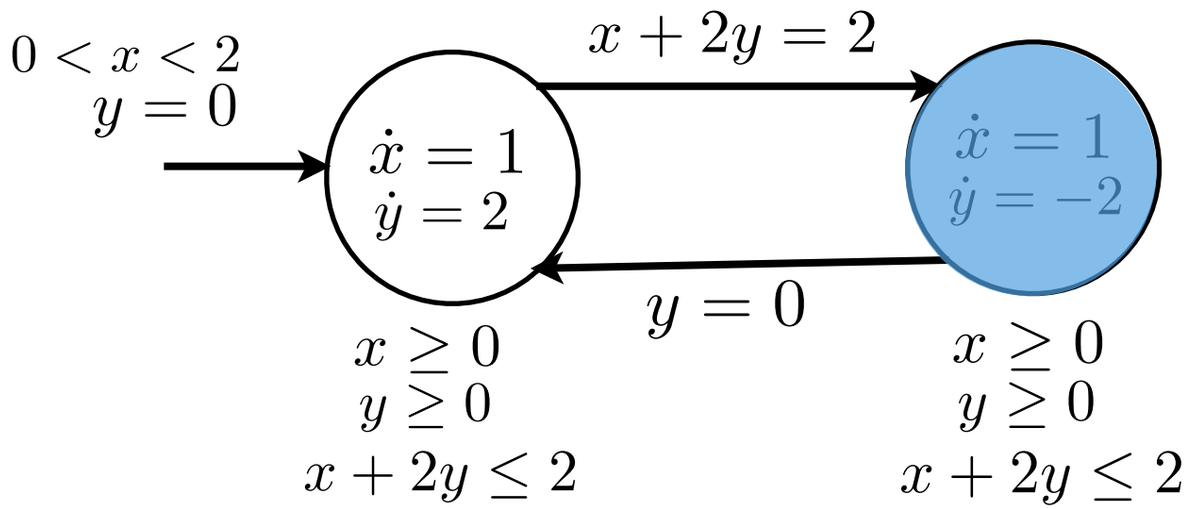

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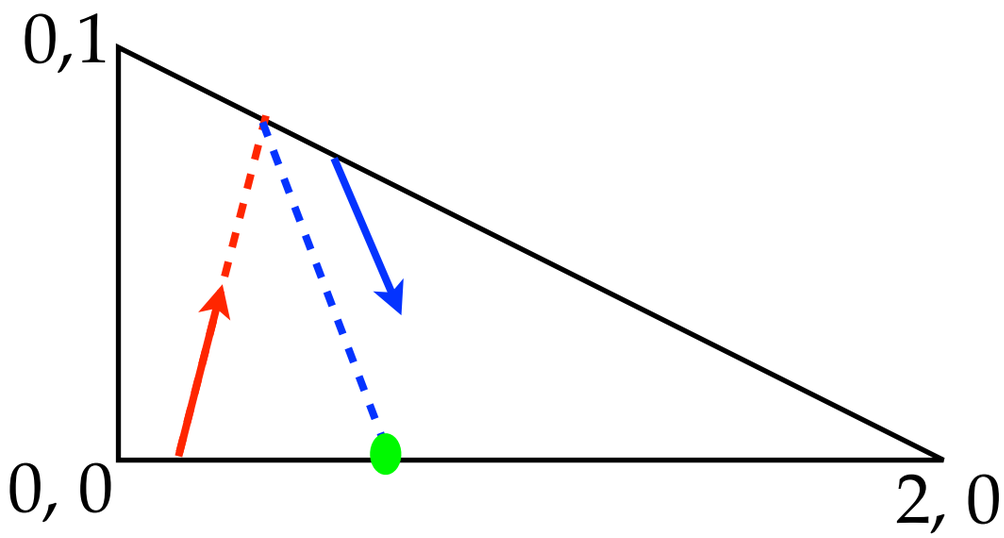

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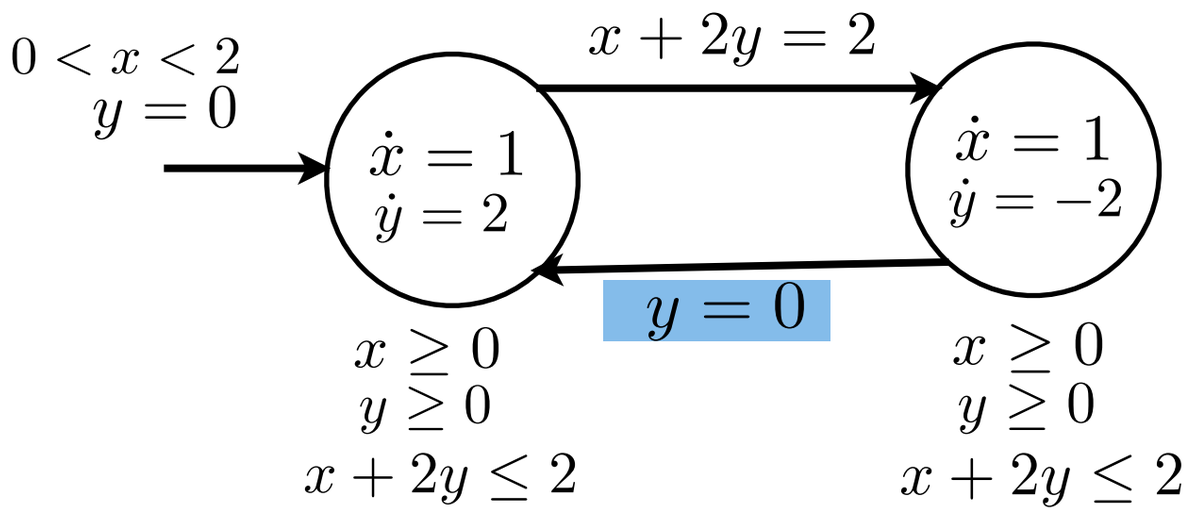

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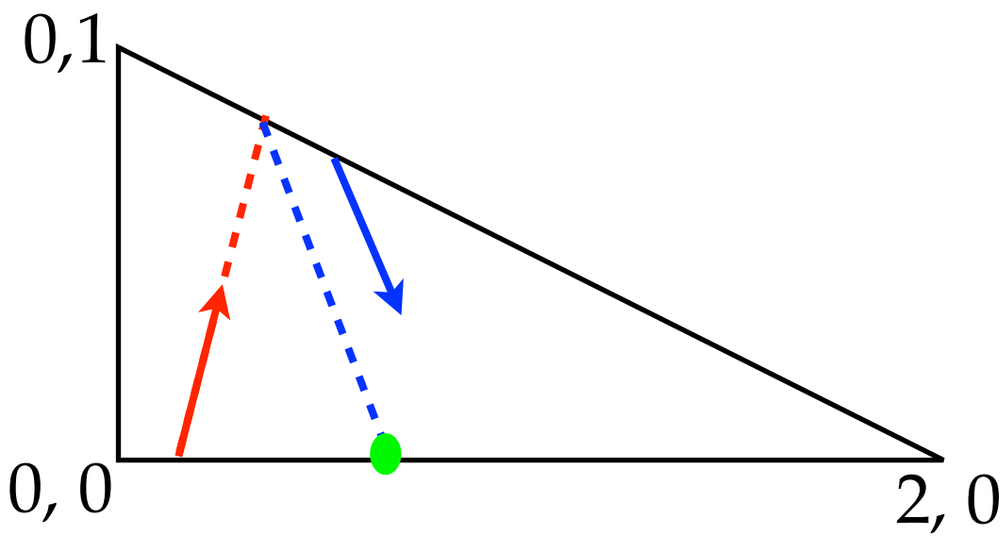

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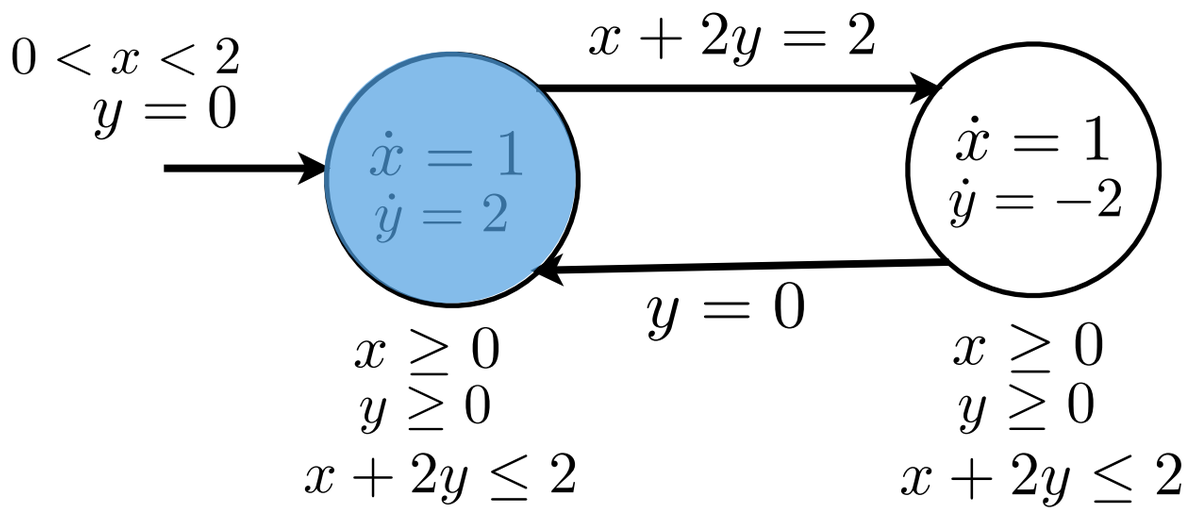

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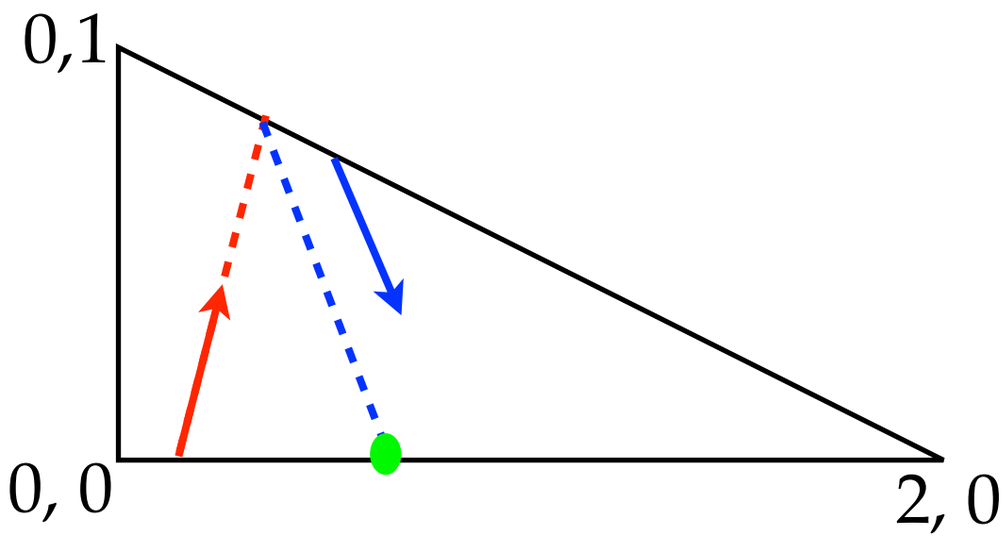

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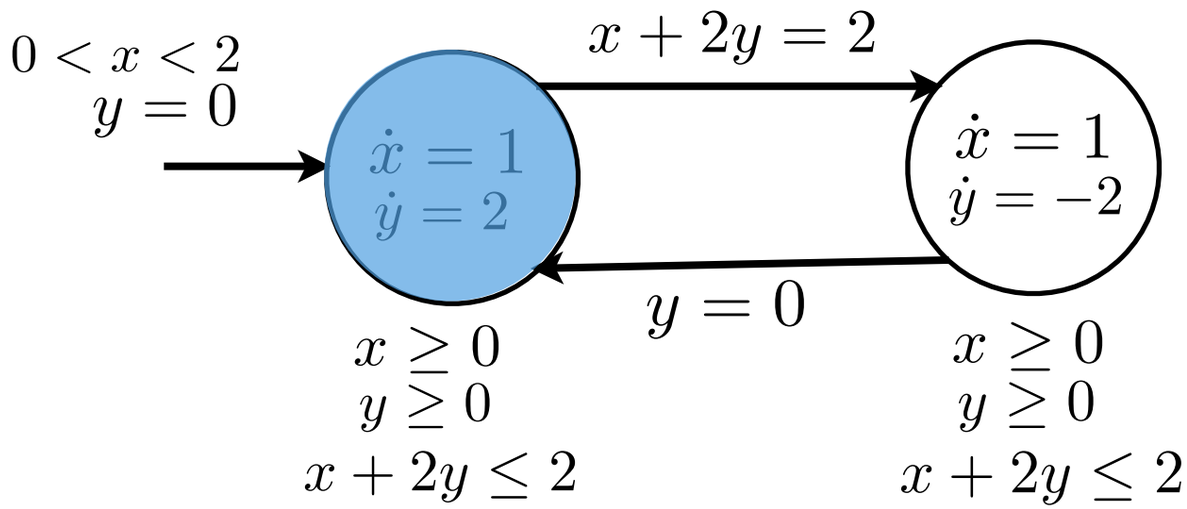

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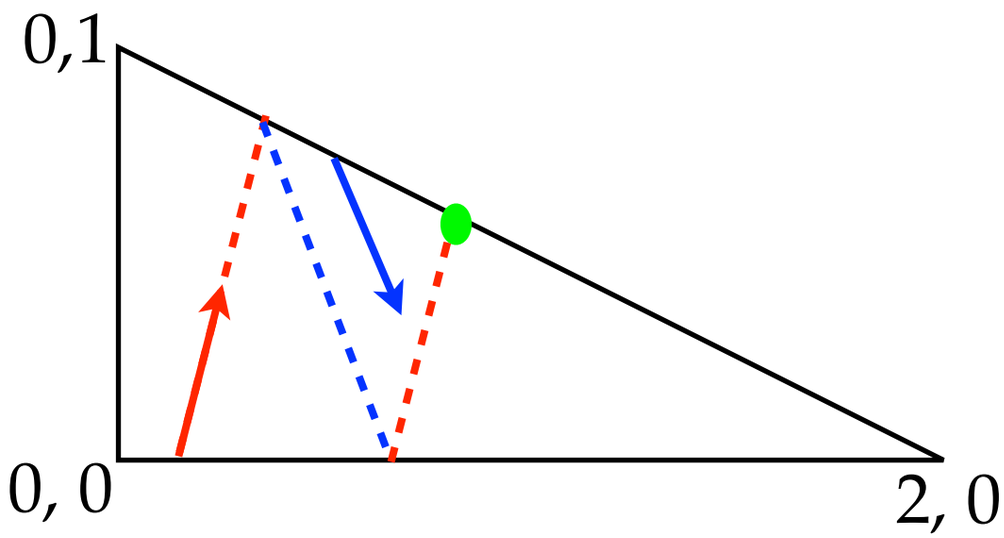
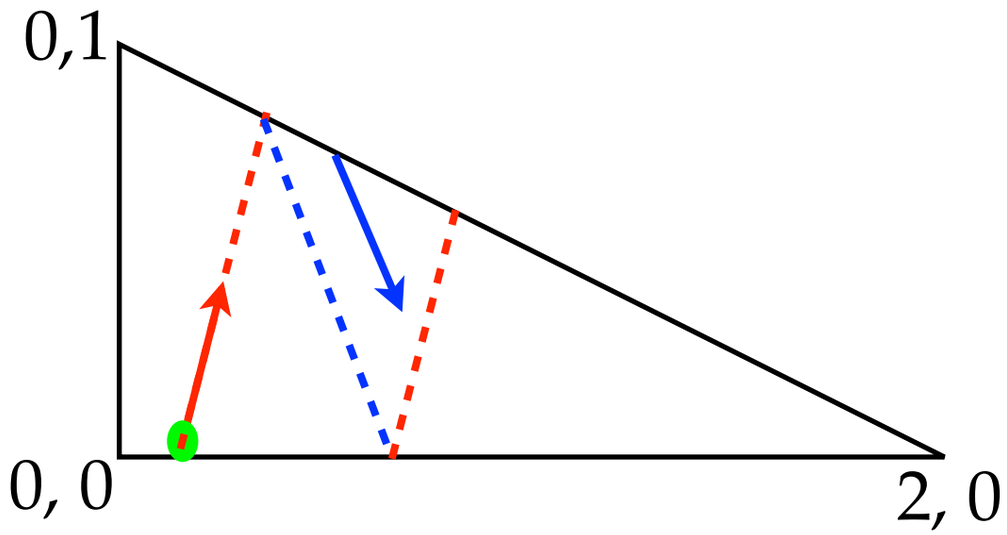


Safety analysis - Reach set computation



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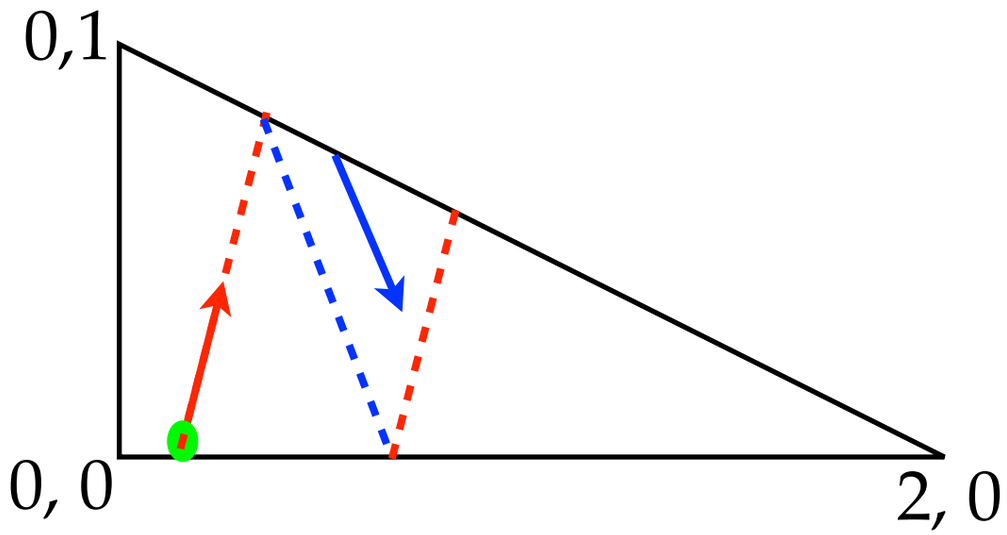

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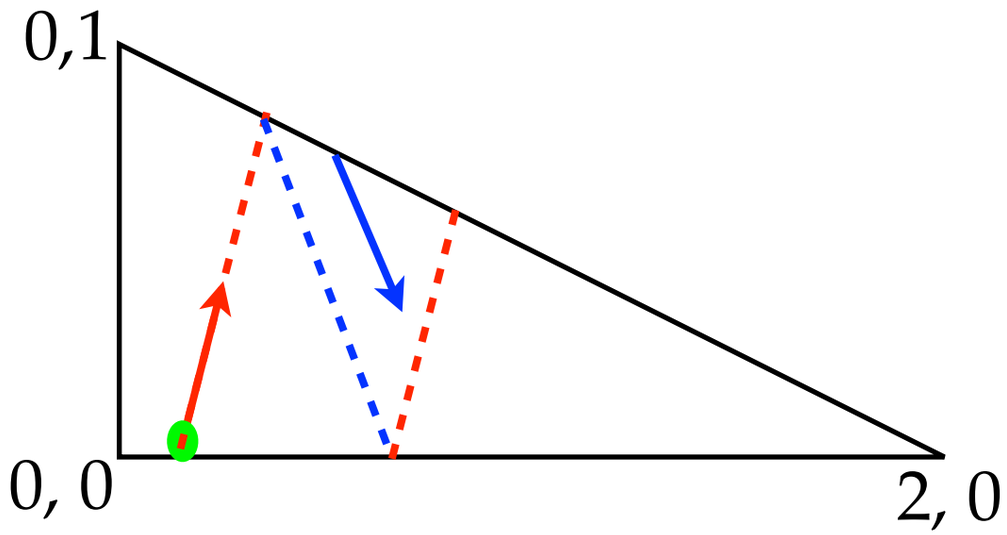

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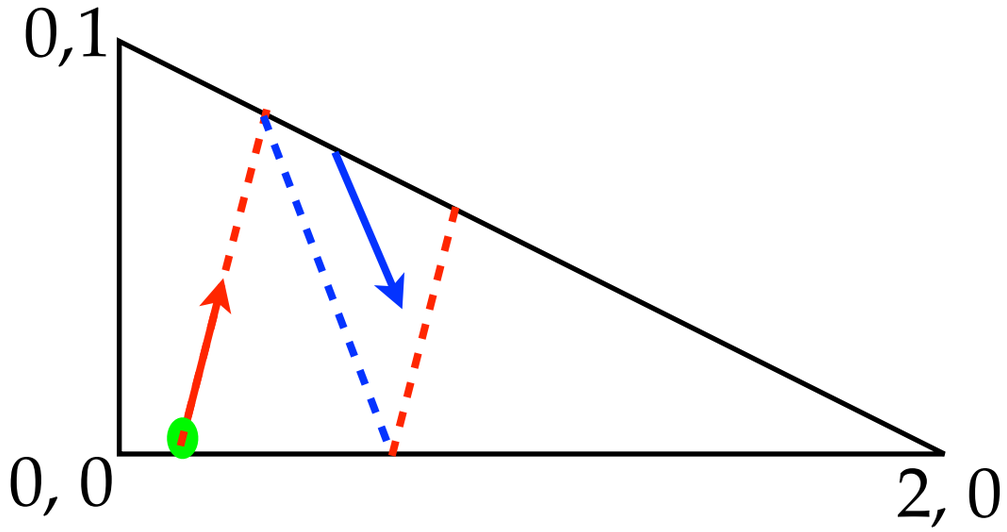

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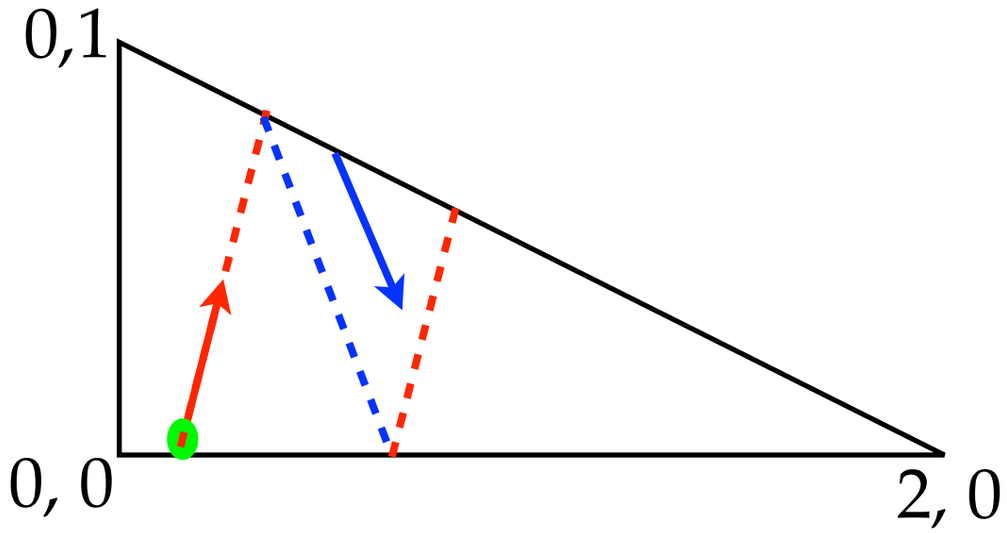

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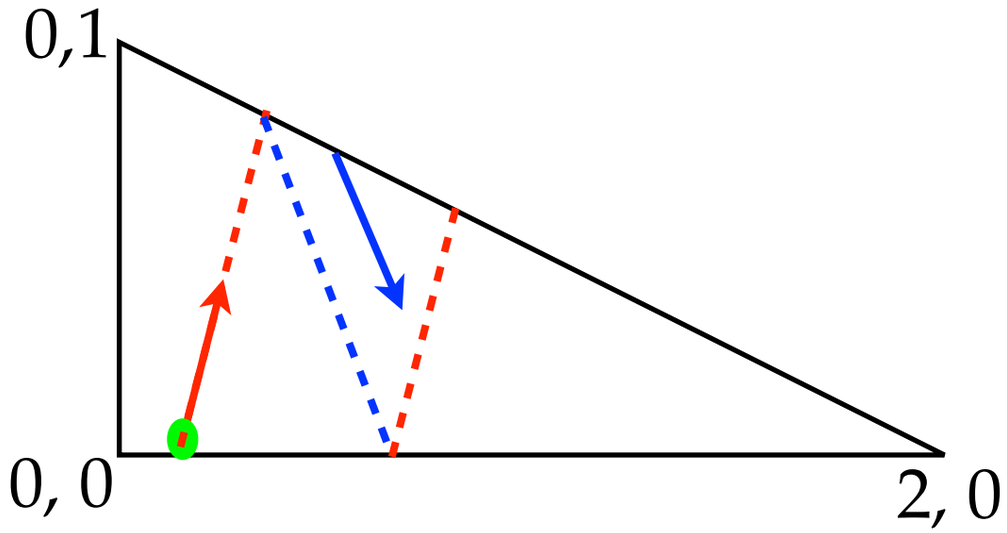

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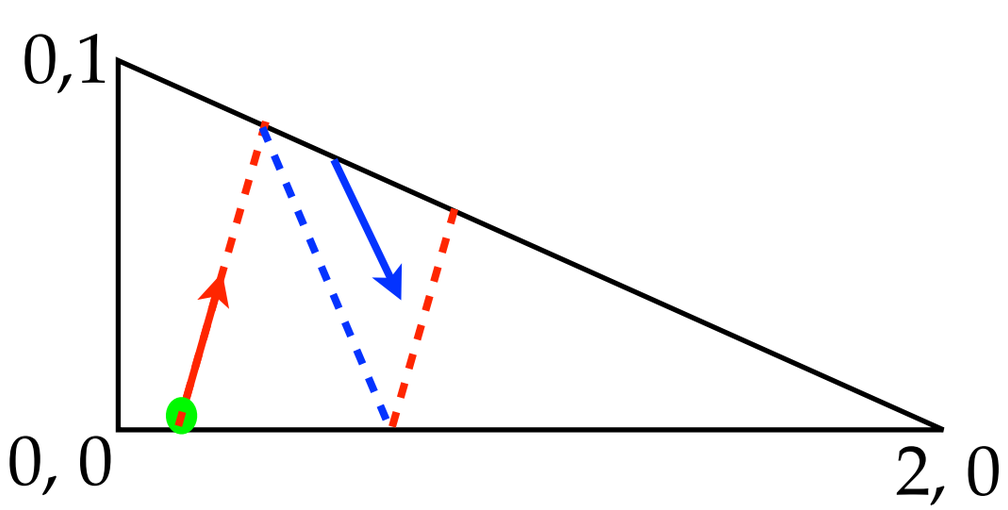

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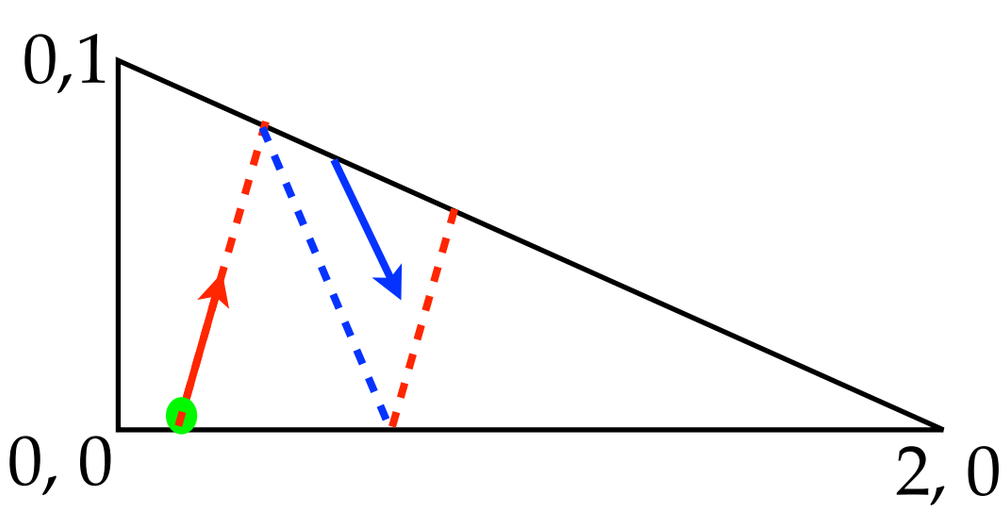

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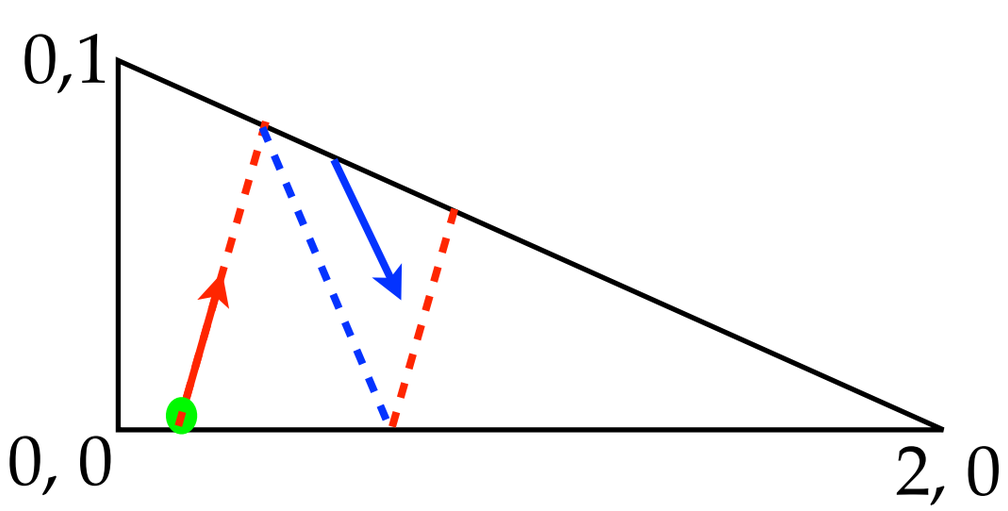

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- Sets represents by polyhedral sets or formulas over first-order logic
- Emptiness checking reduces to satisfiability problem of the logic
    - Satisfiability of first order logic with addition and multiplication is decidable.

A more complex class

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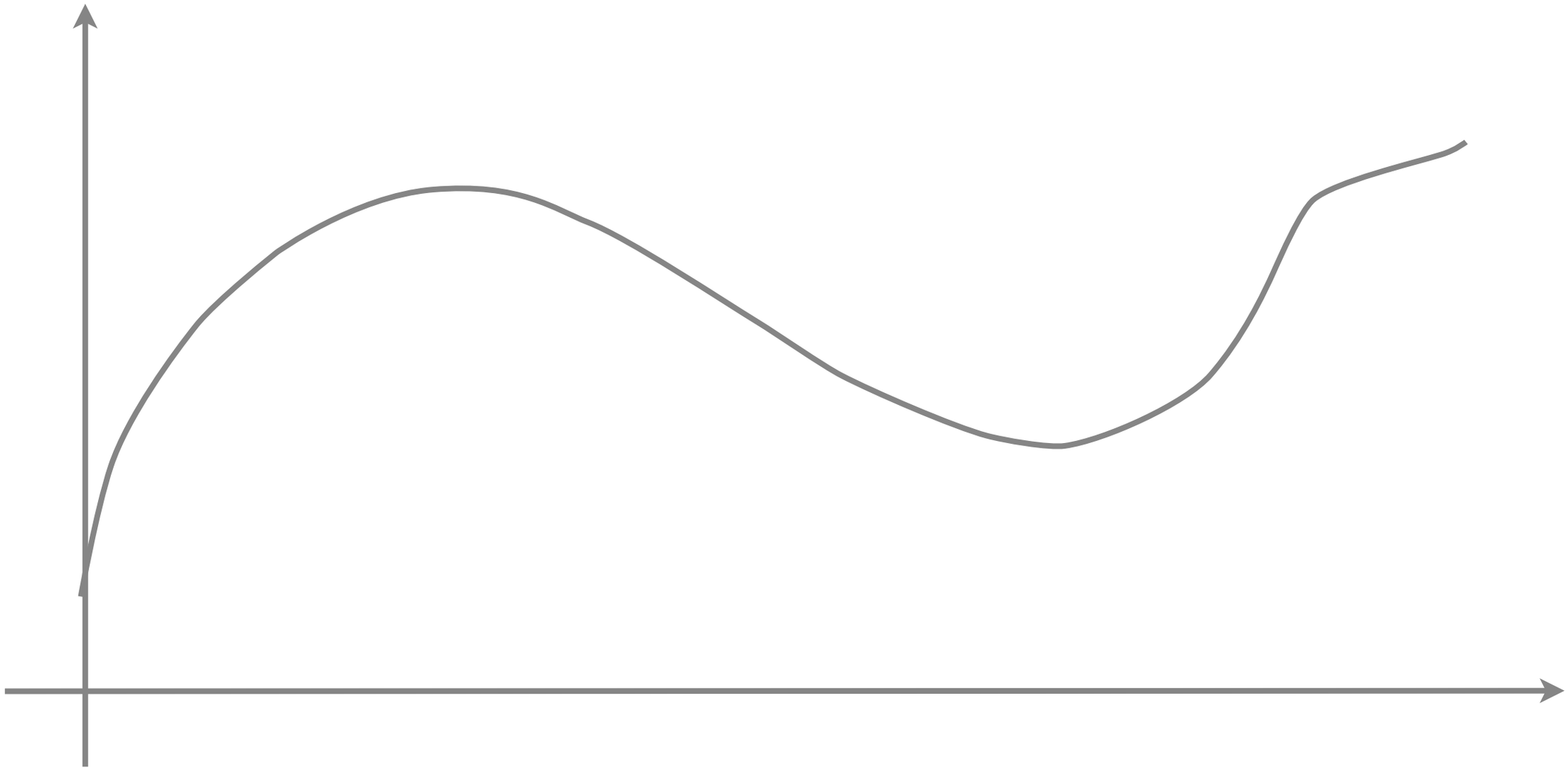

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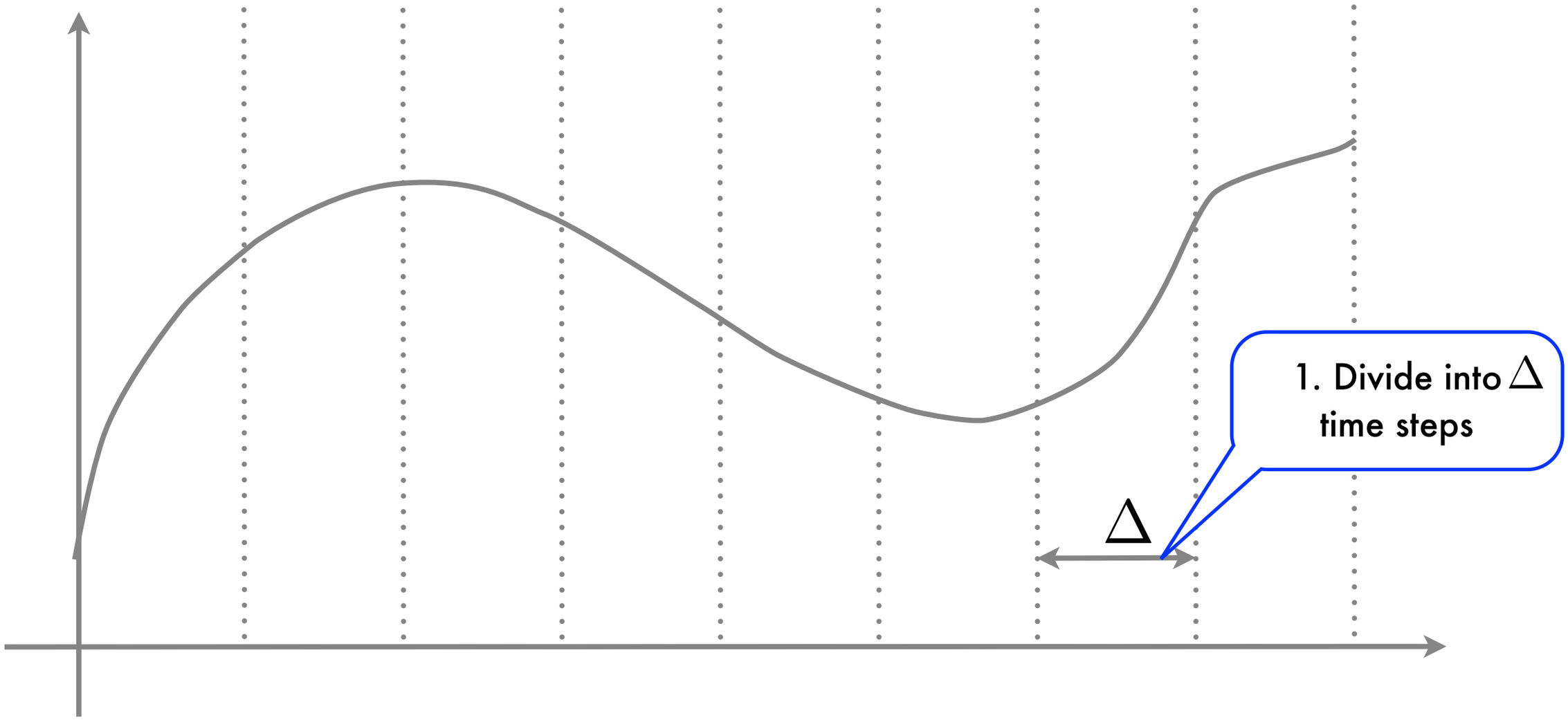

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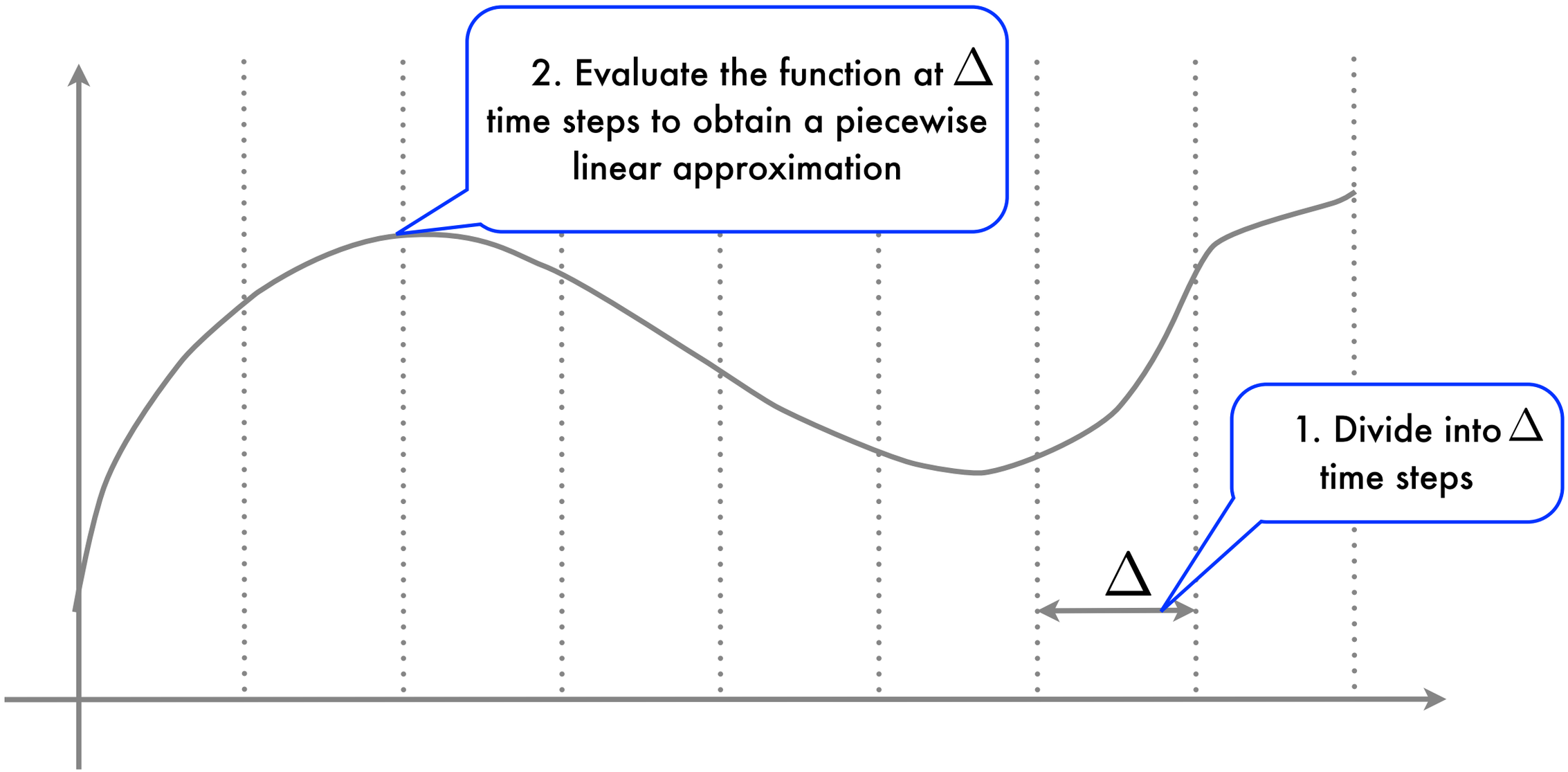

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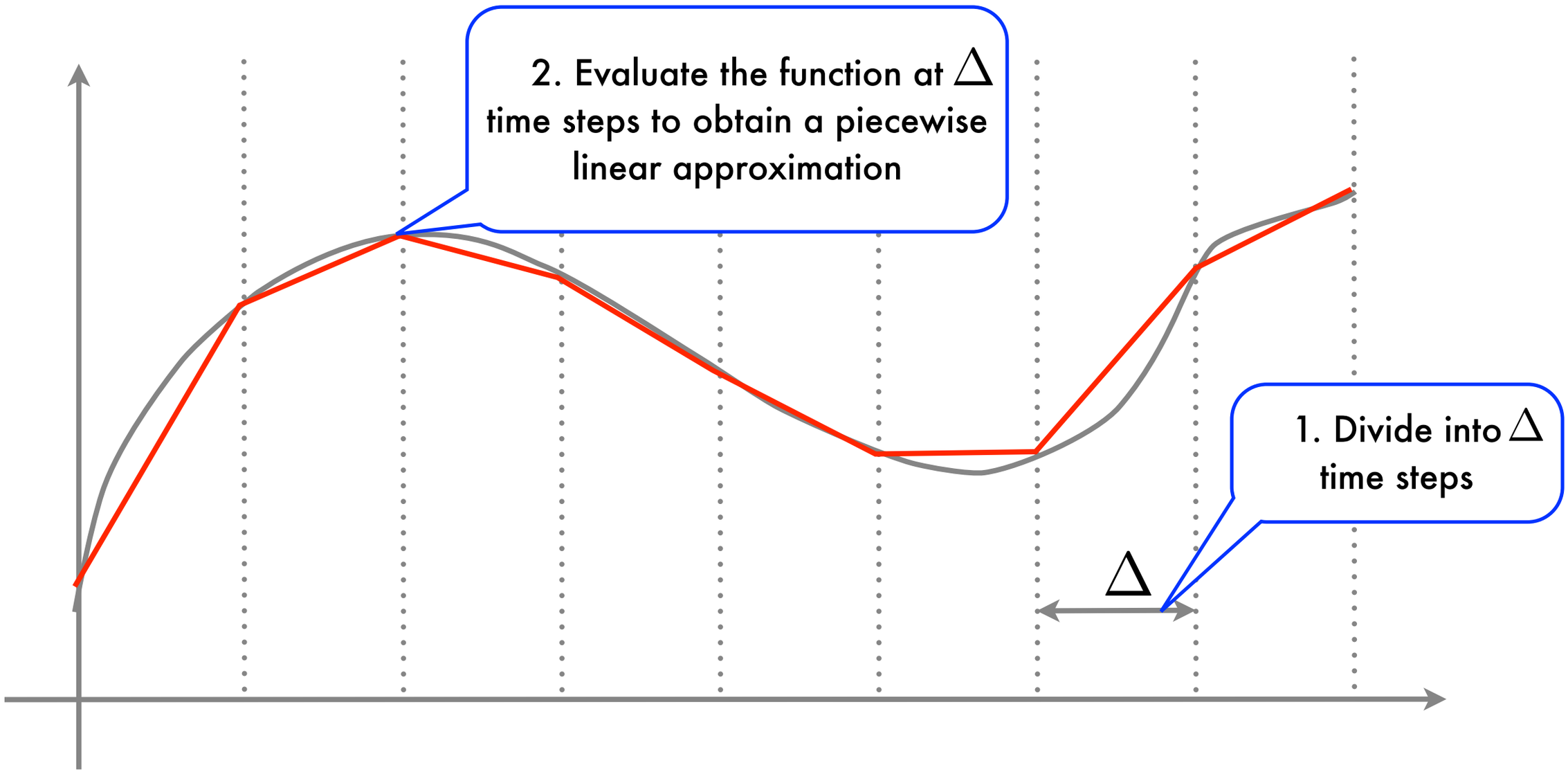

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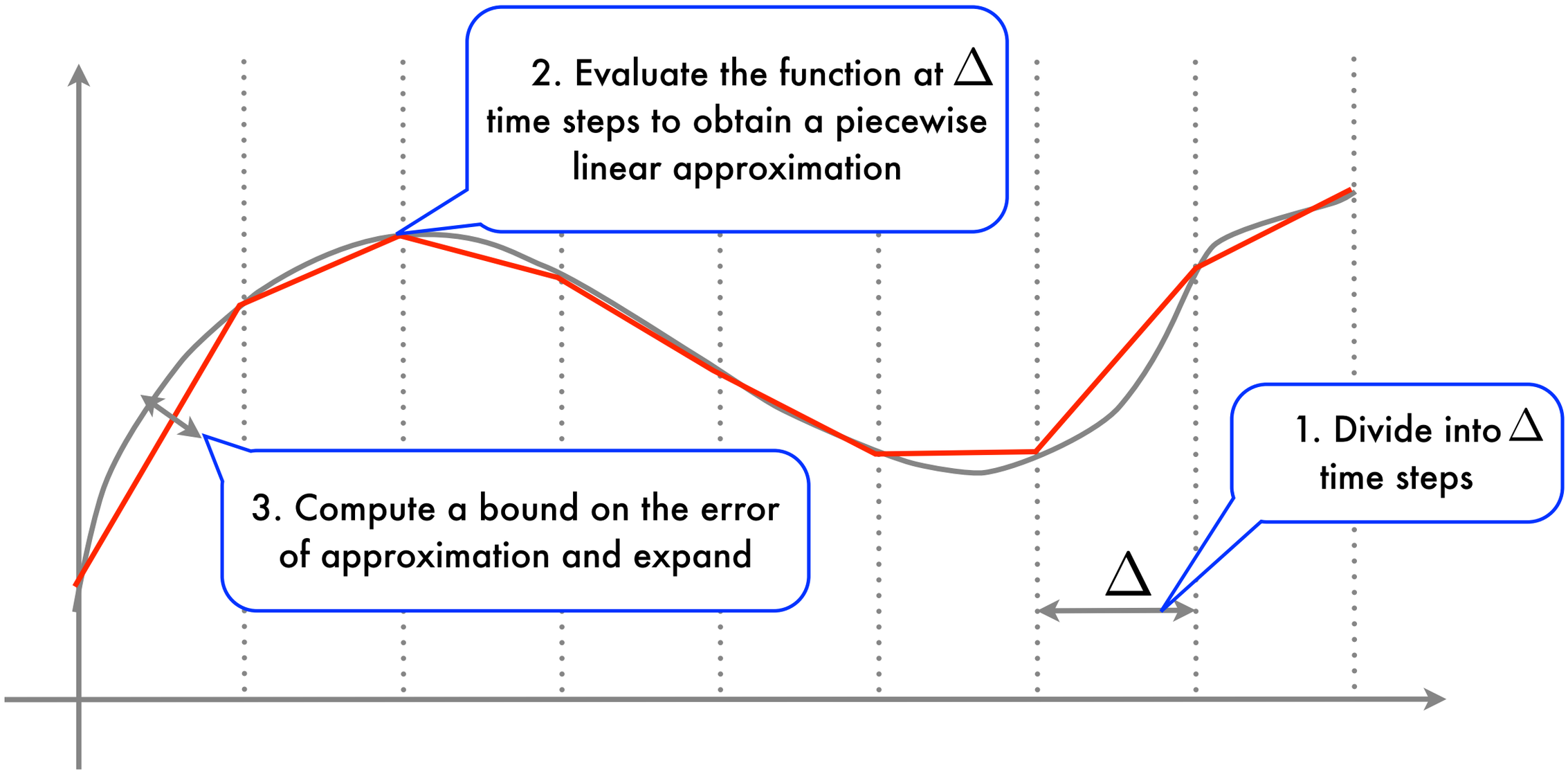

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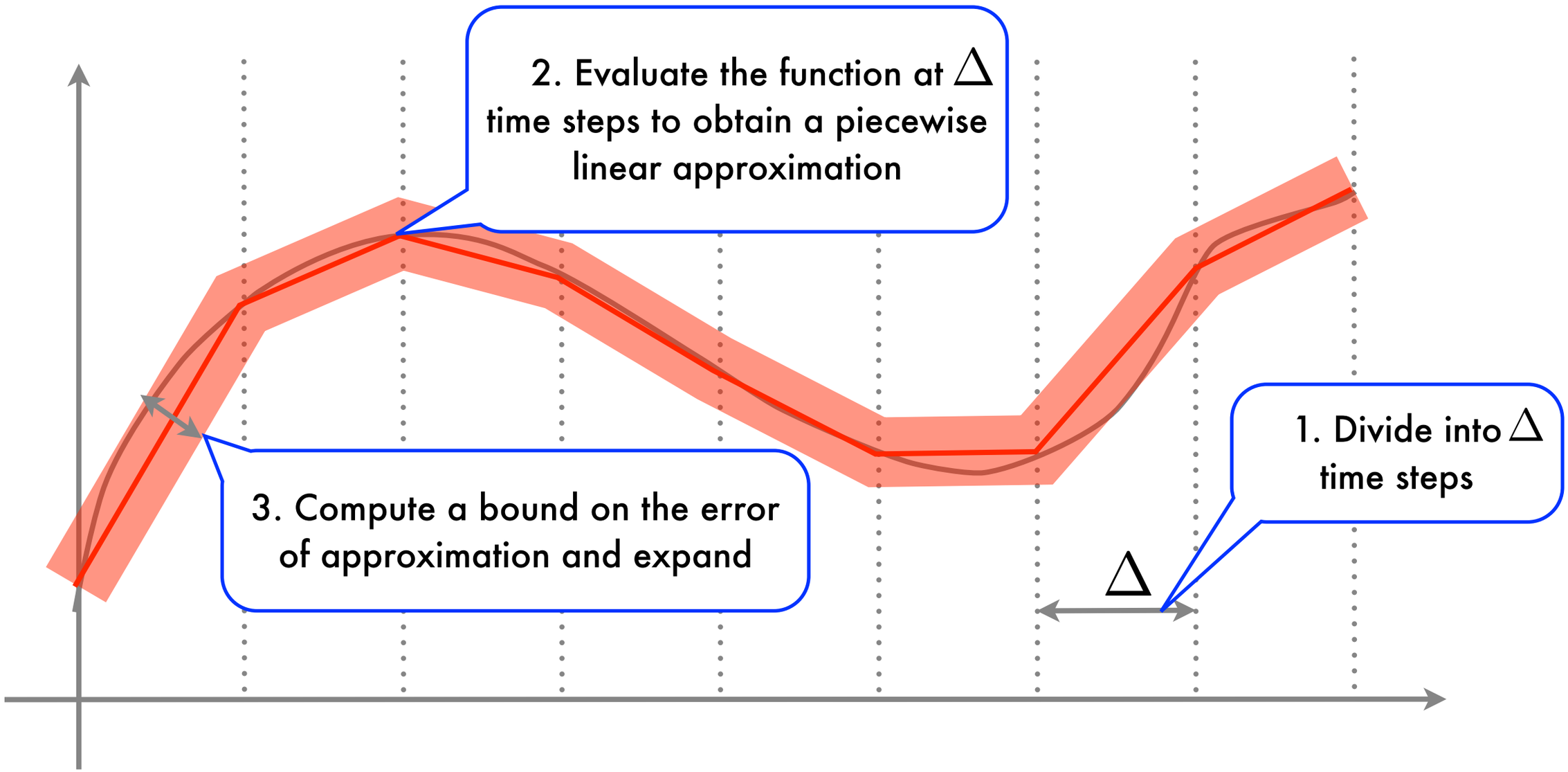

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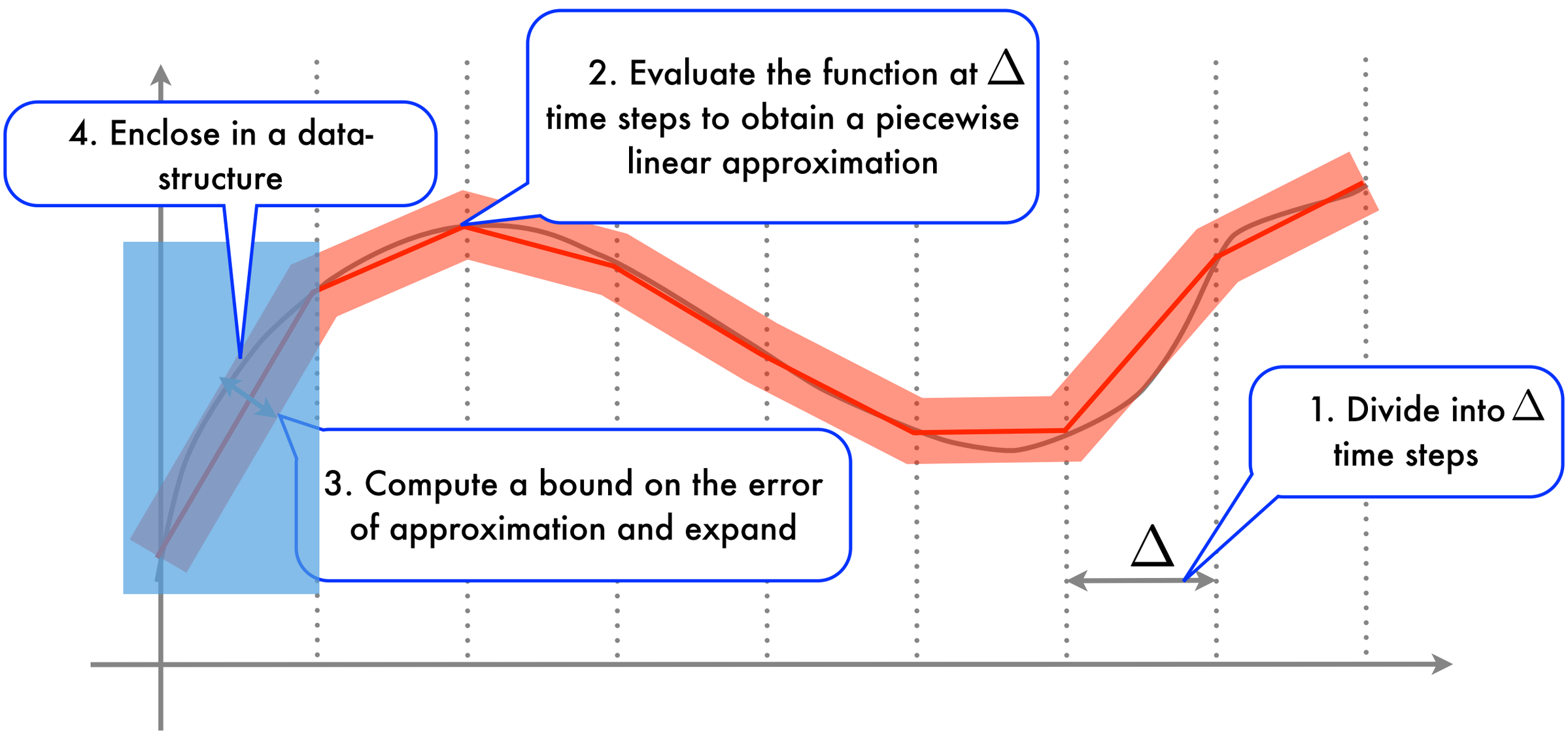

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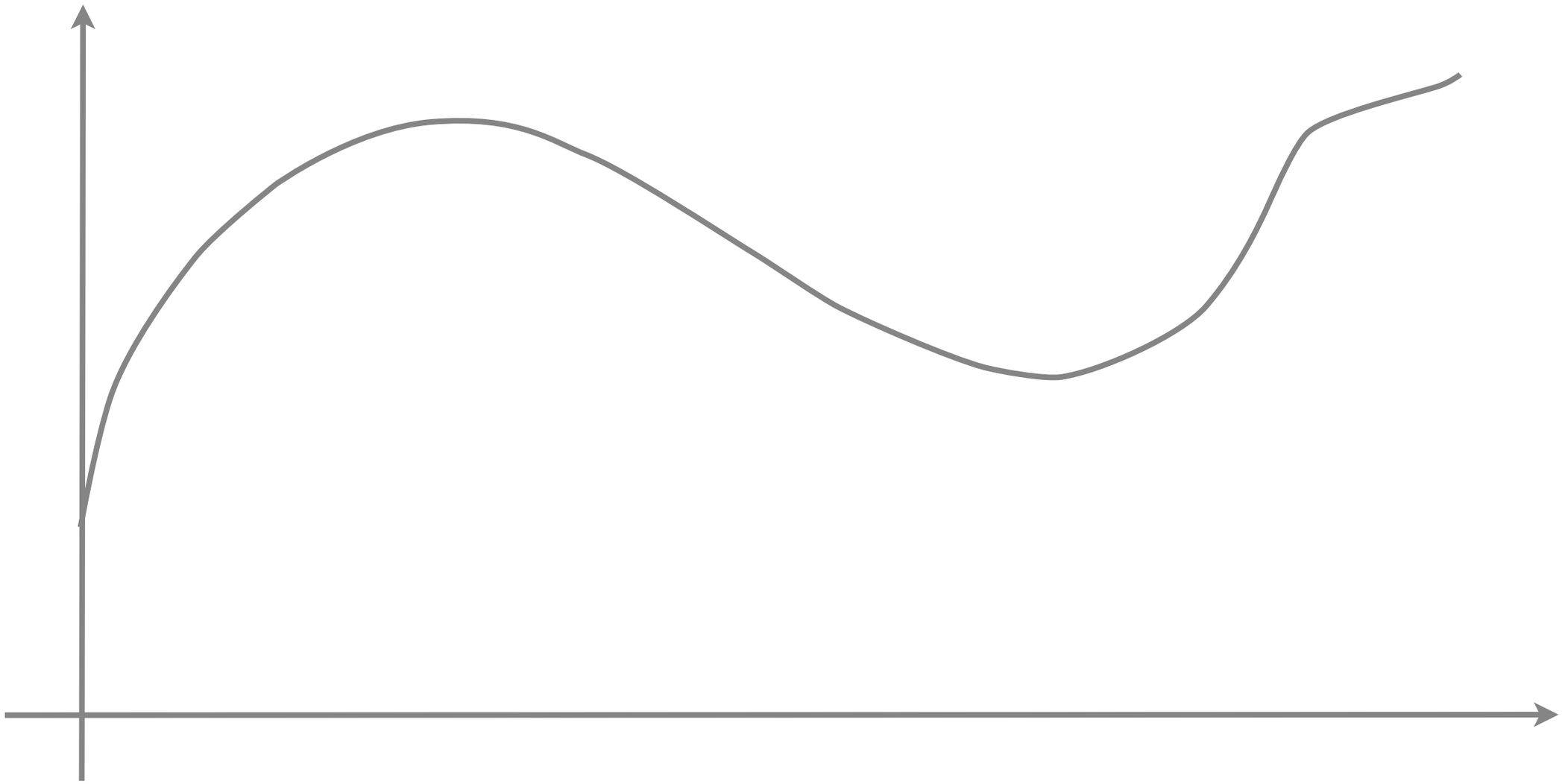

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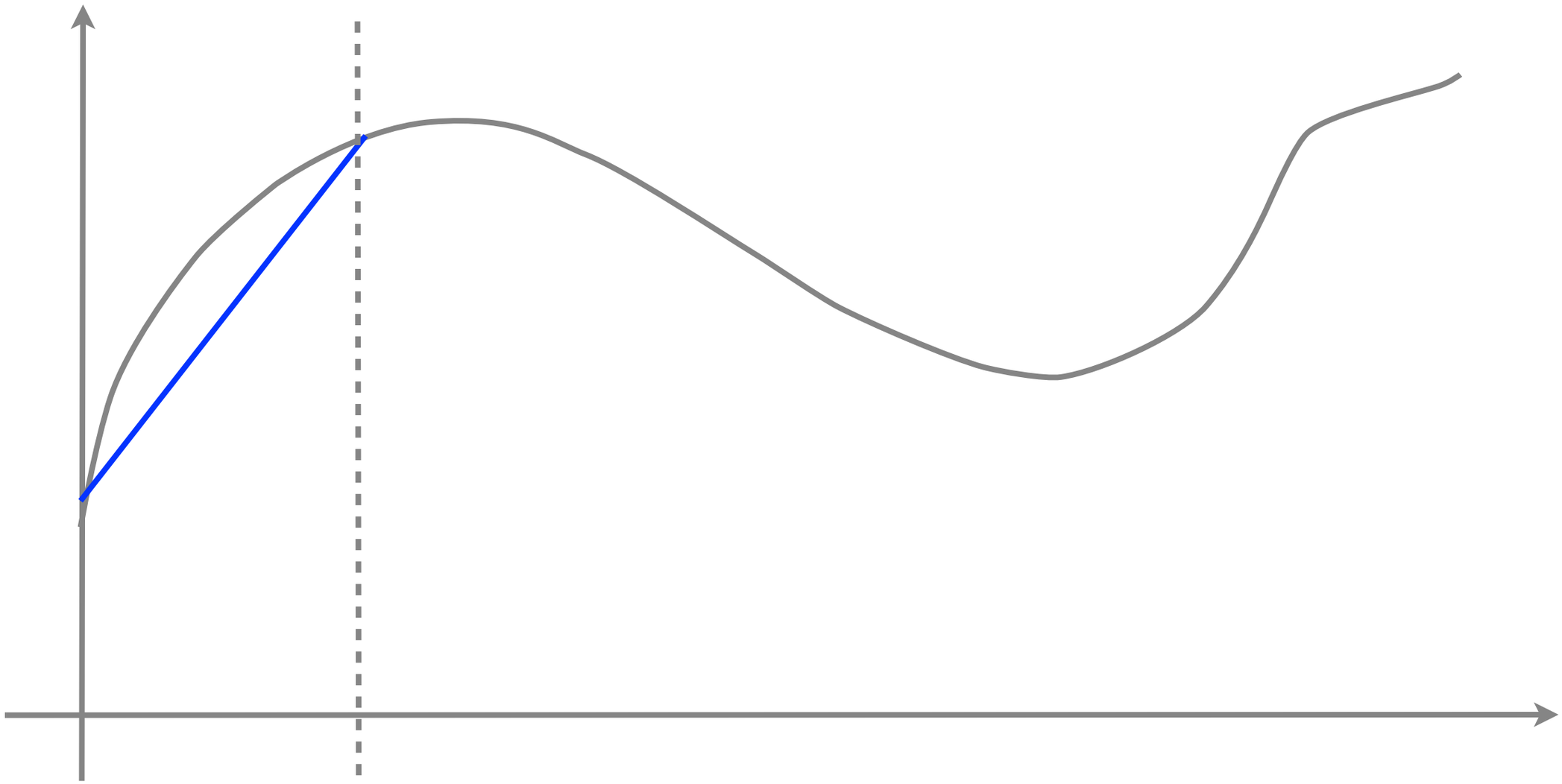

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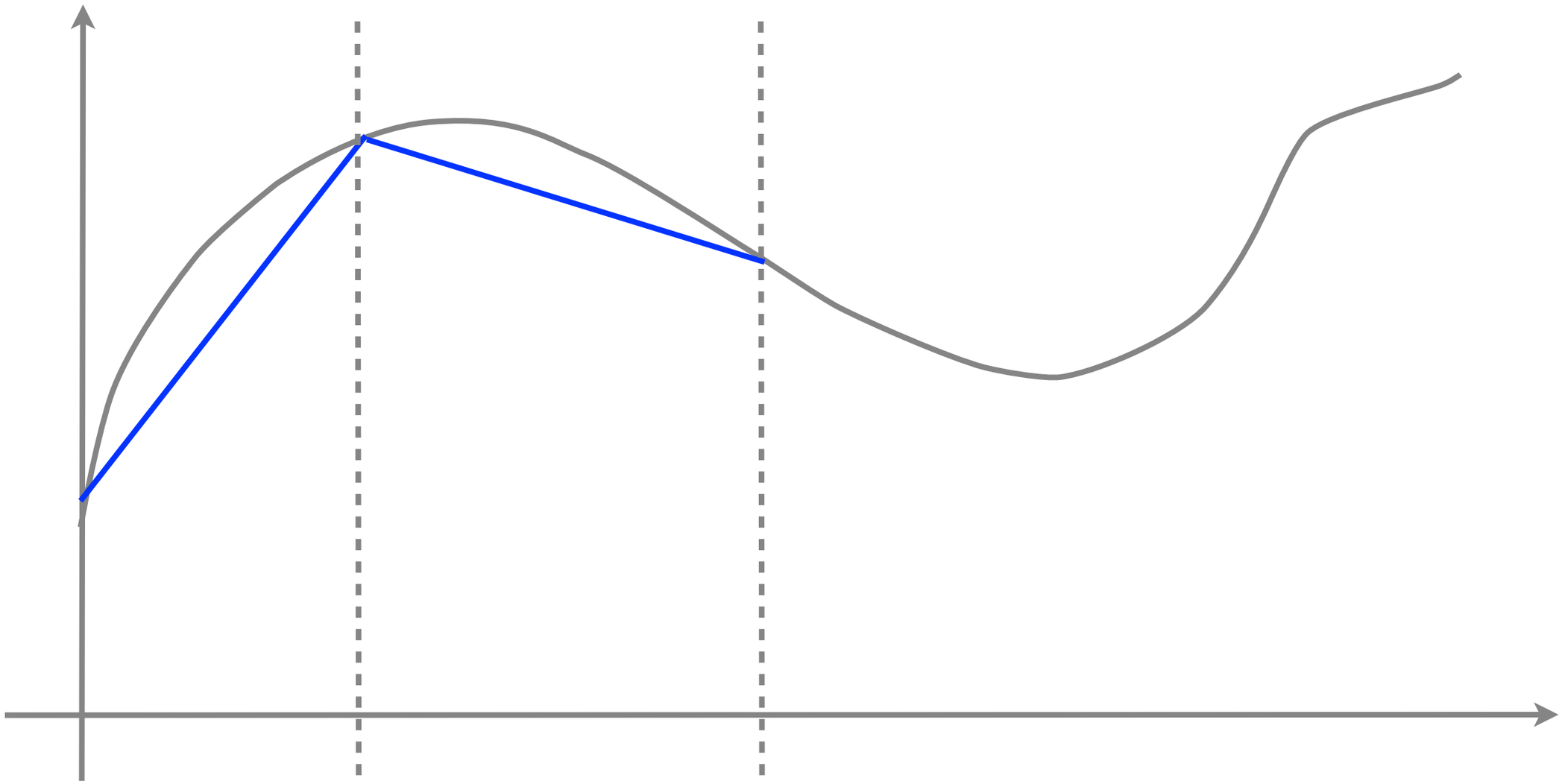

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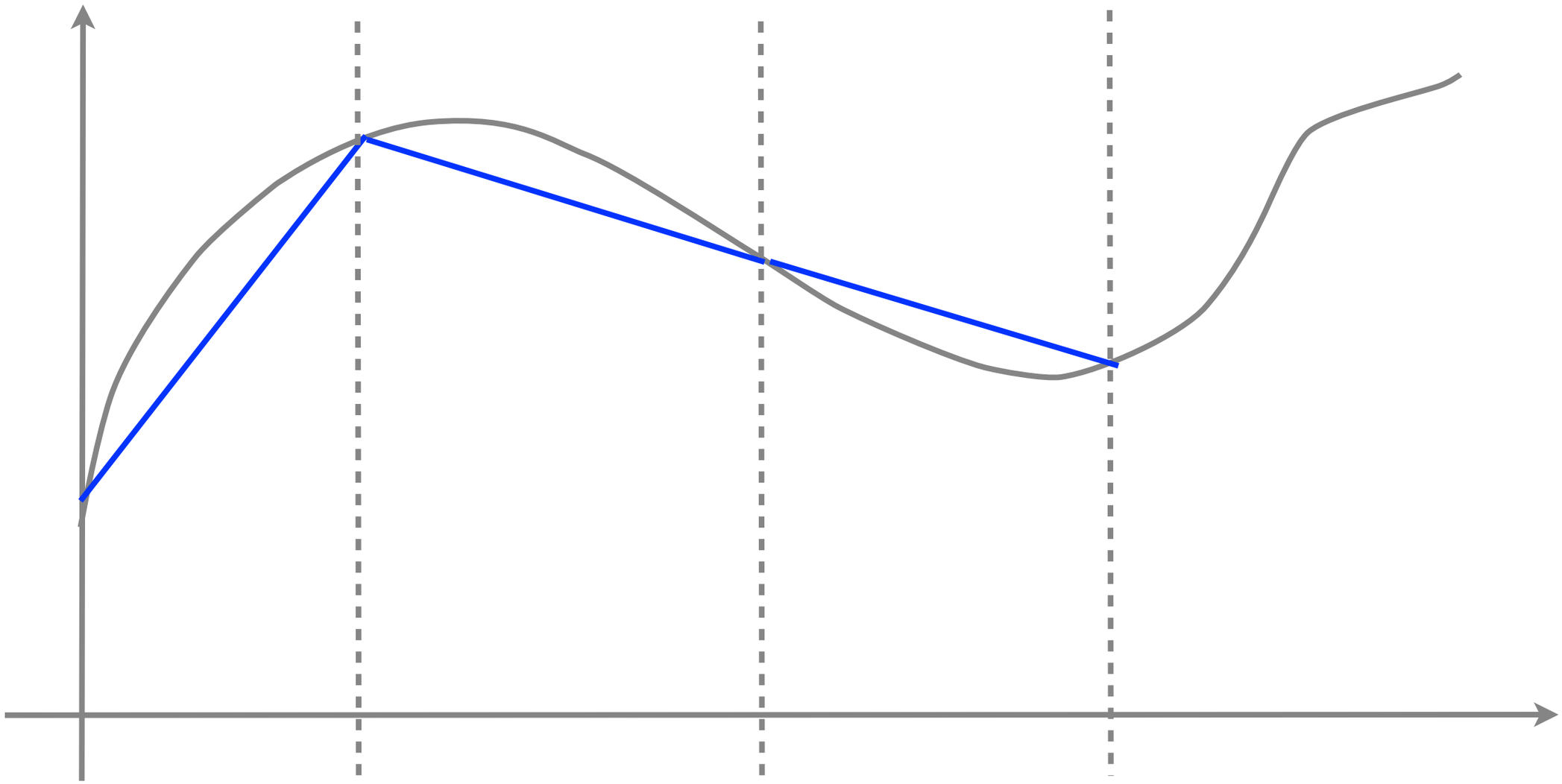

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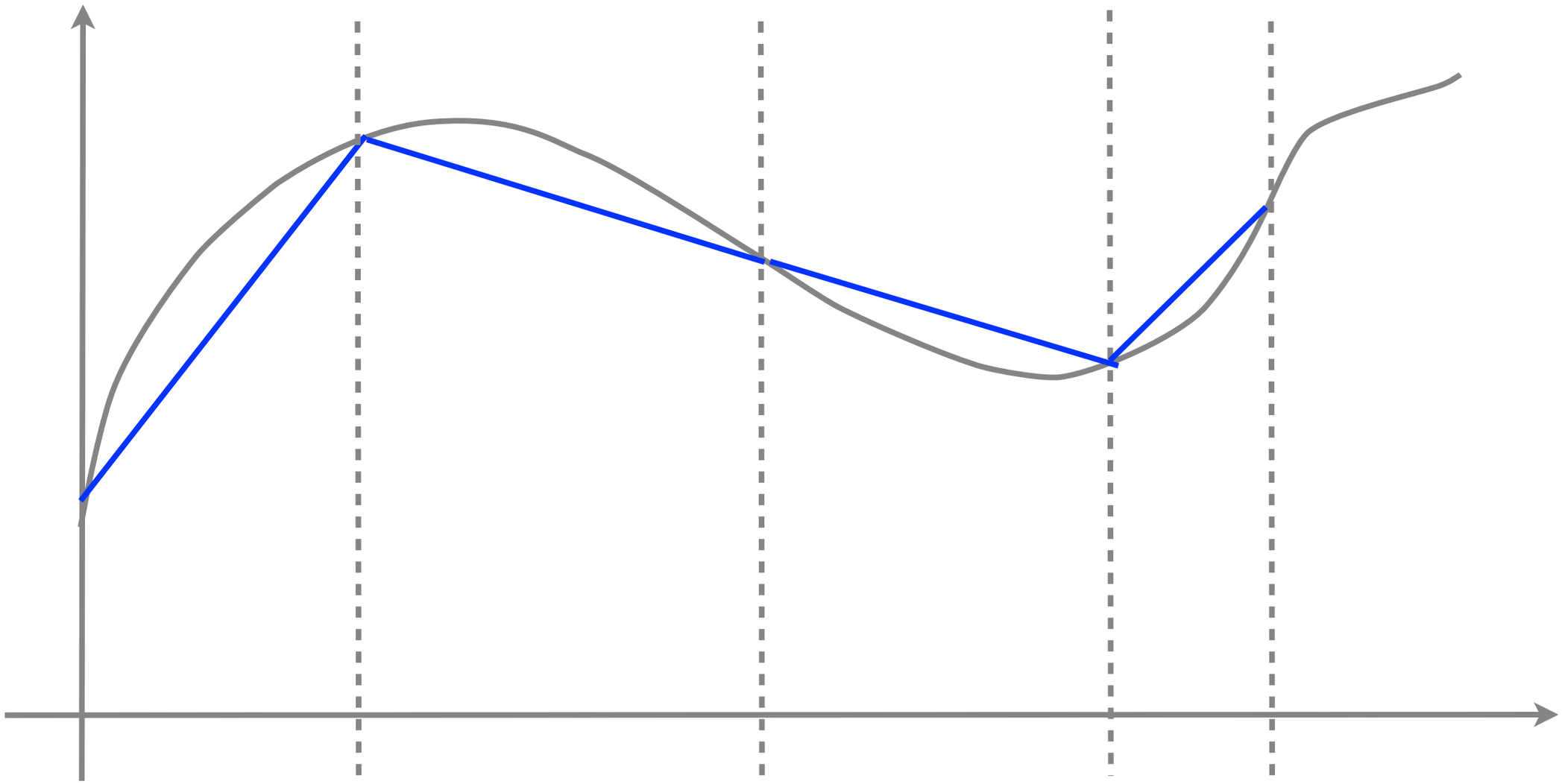

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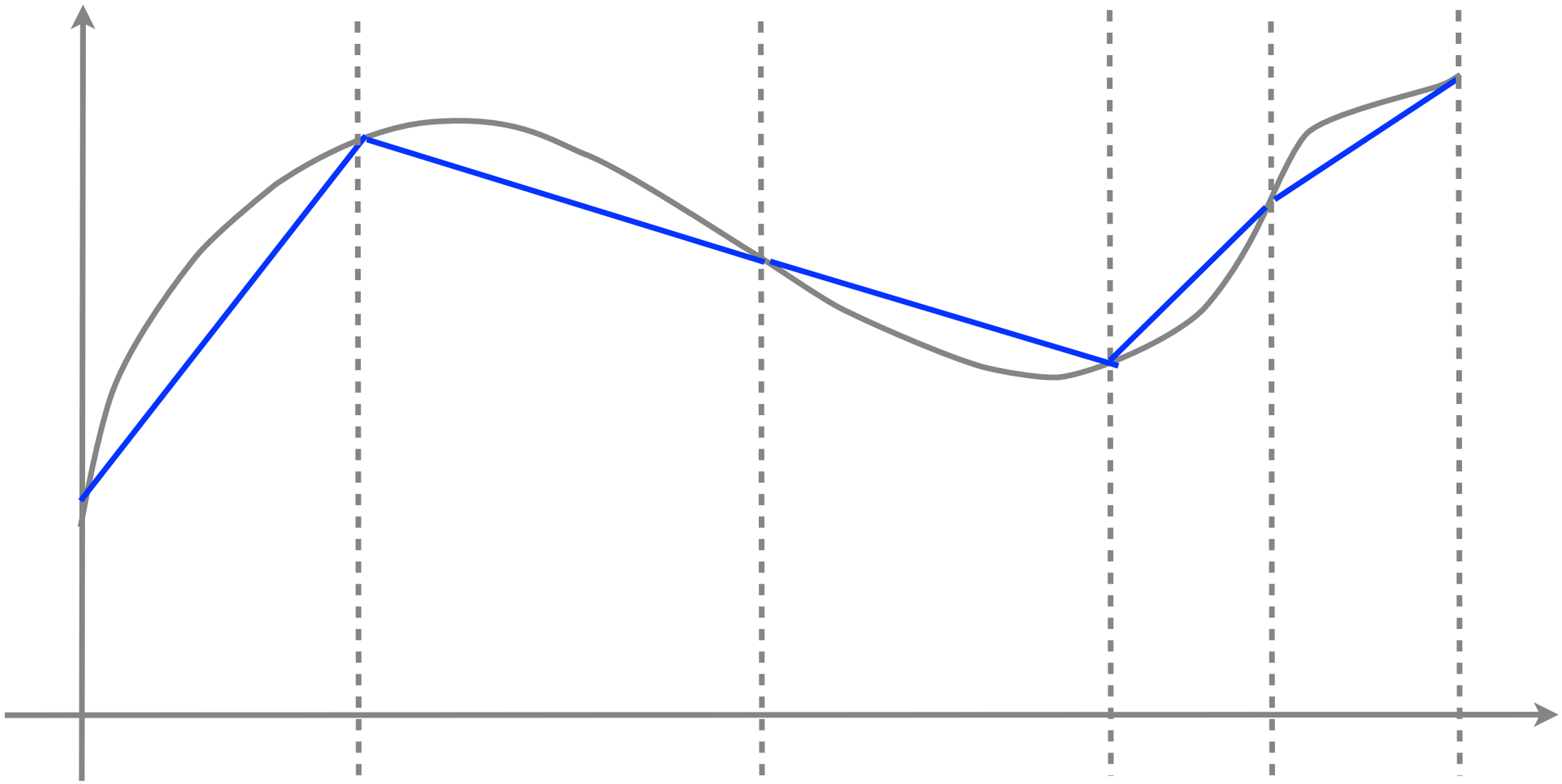

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- Hybridization [Puri, Borkar, Varaiya], [Asarin,Dang,Girard]
  - Finite partition of the state-space.
  - Approximate dynamics using the right hand side of the differential equation.

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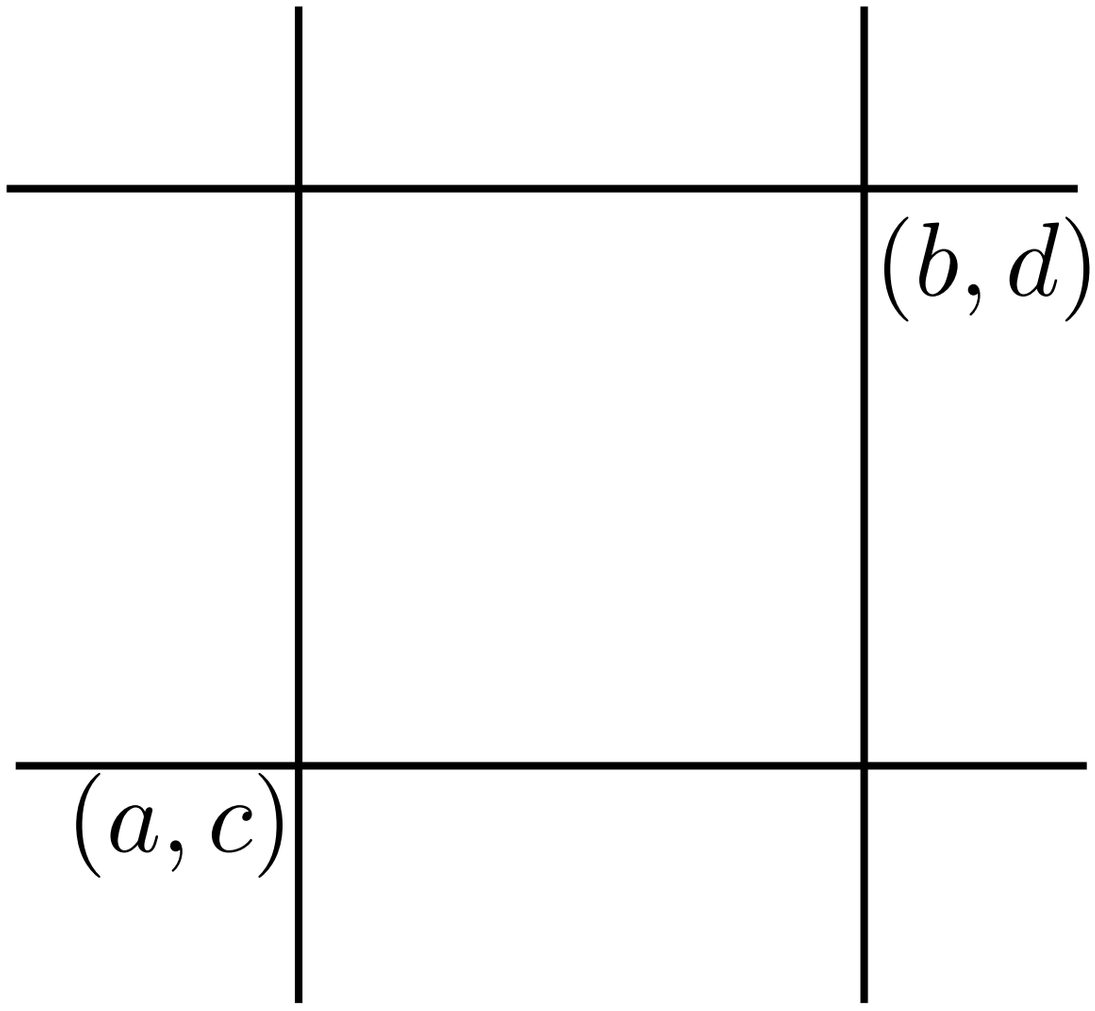

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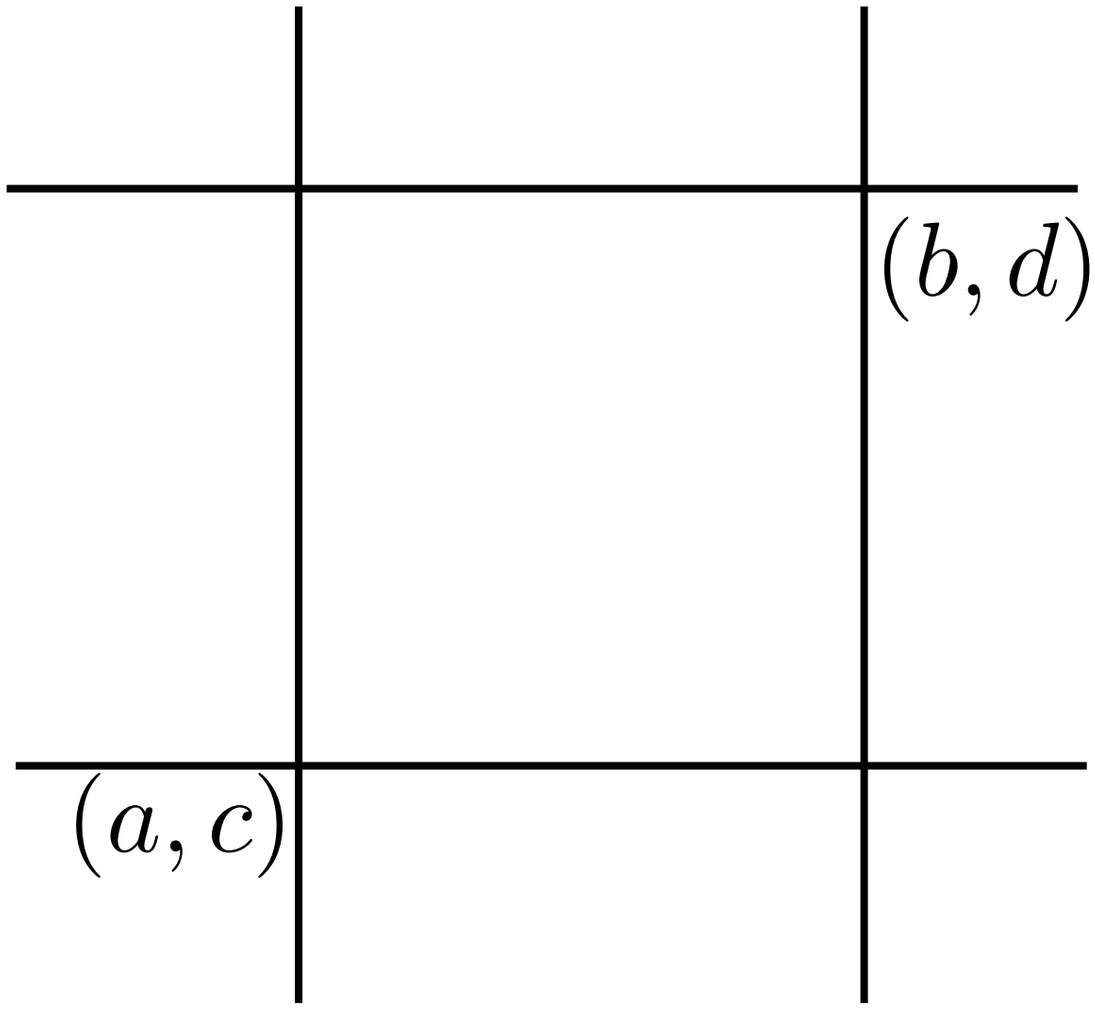

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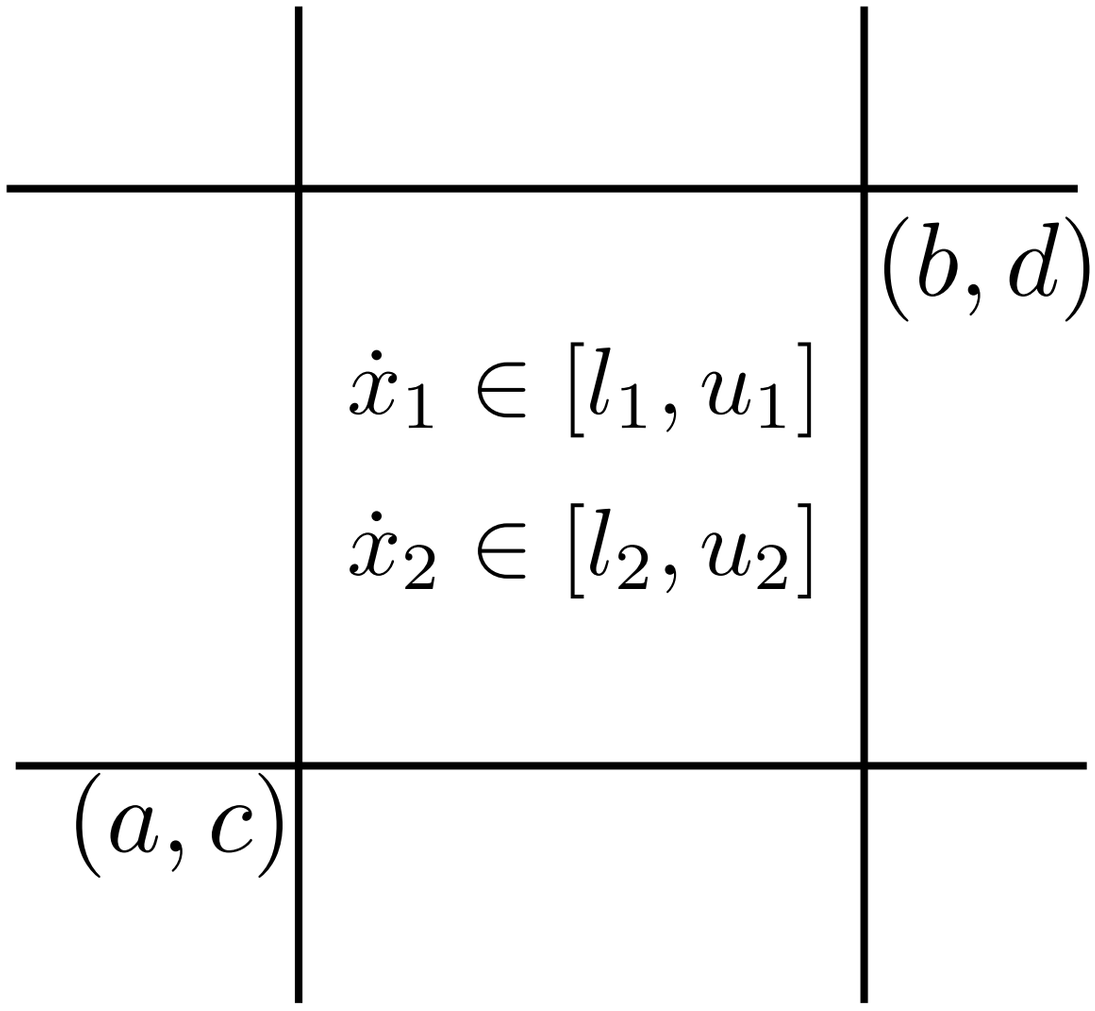
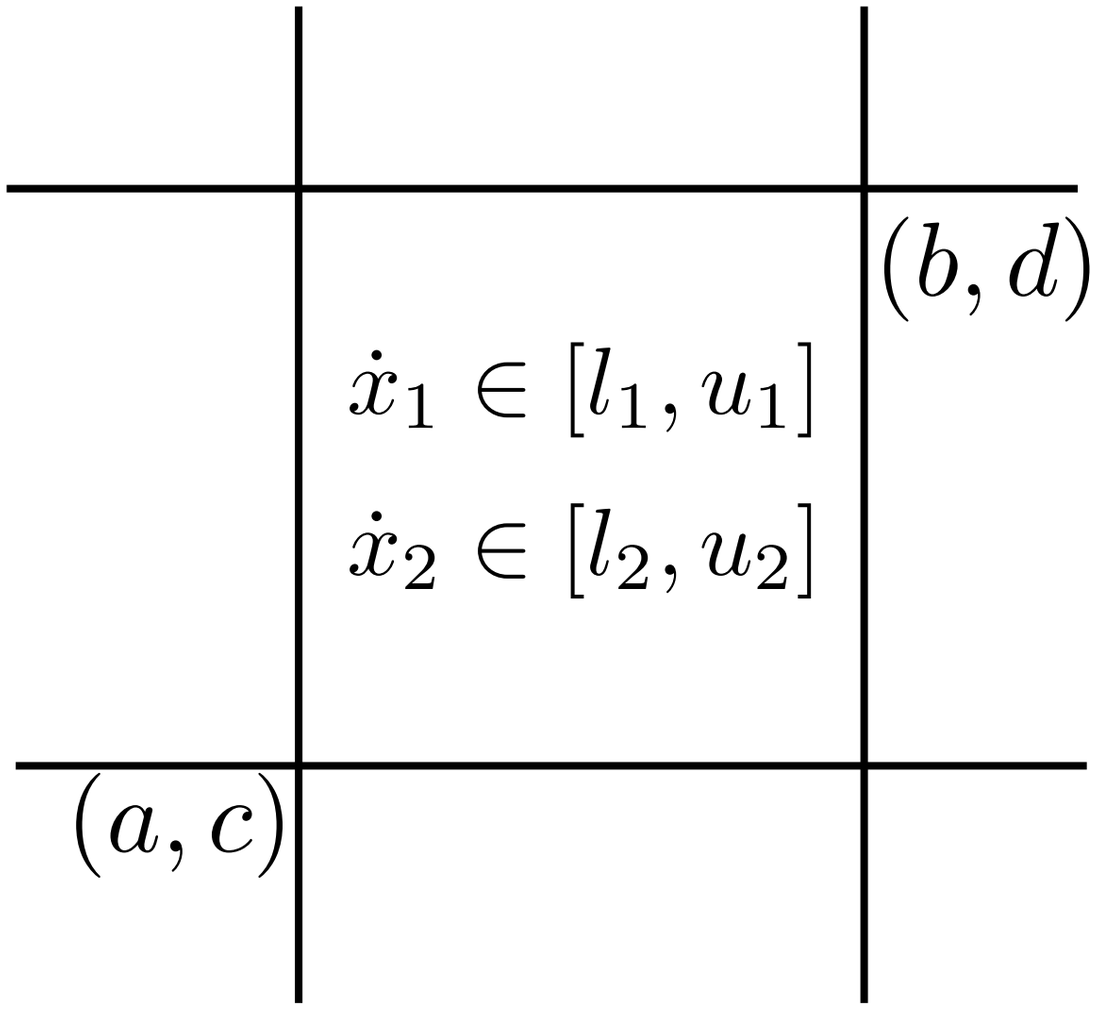


Hybridization - Rectangular approximation

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

Find a rectangular approximation of $f(x)$ in each cell



$$Maximize f_1(x_1, x_2)$$

$$a \leq x_1 \leq b$$

$$c \leq x_2 \leq d$$

Hybridization - Linear Approximation

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

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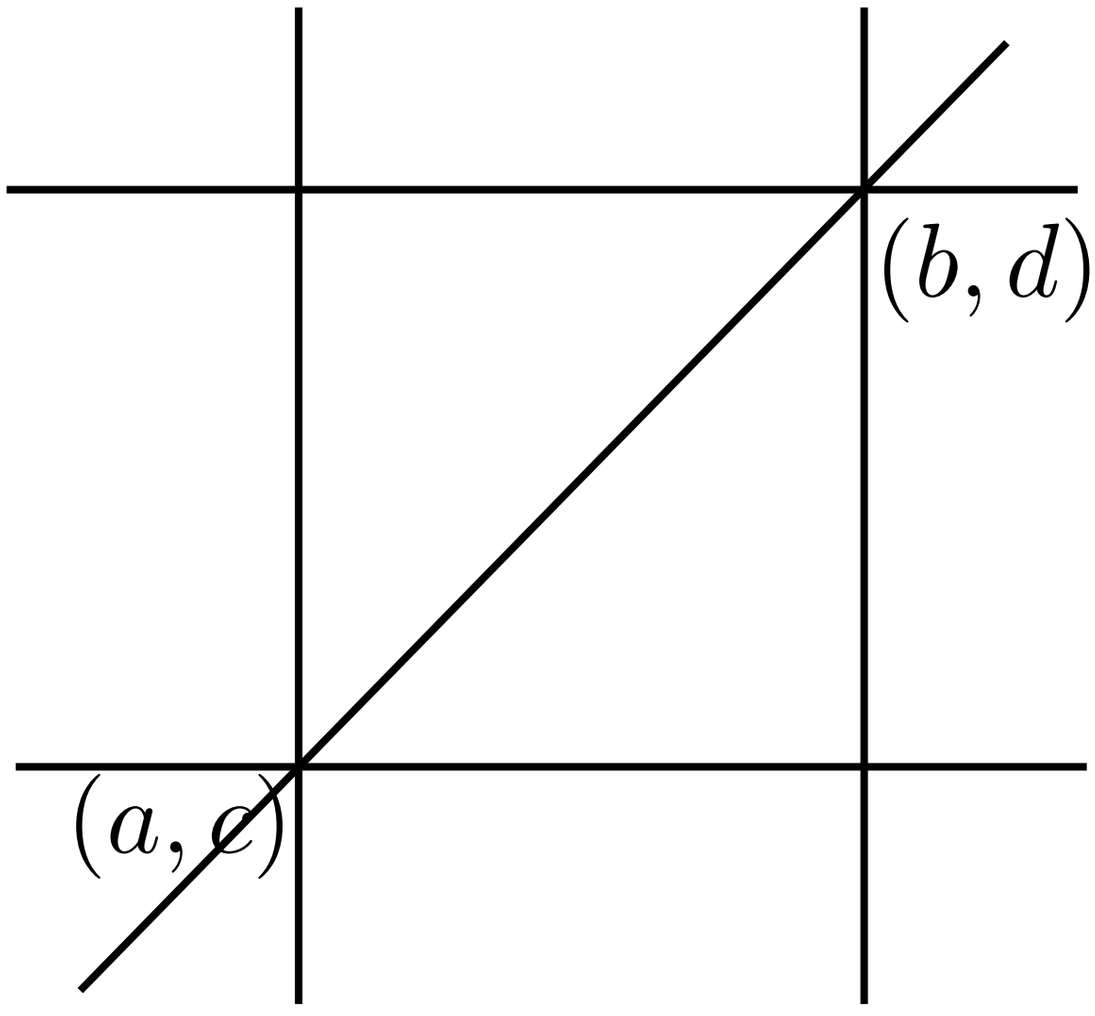
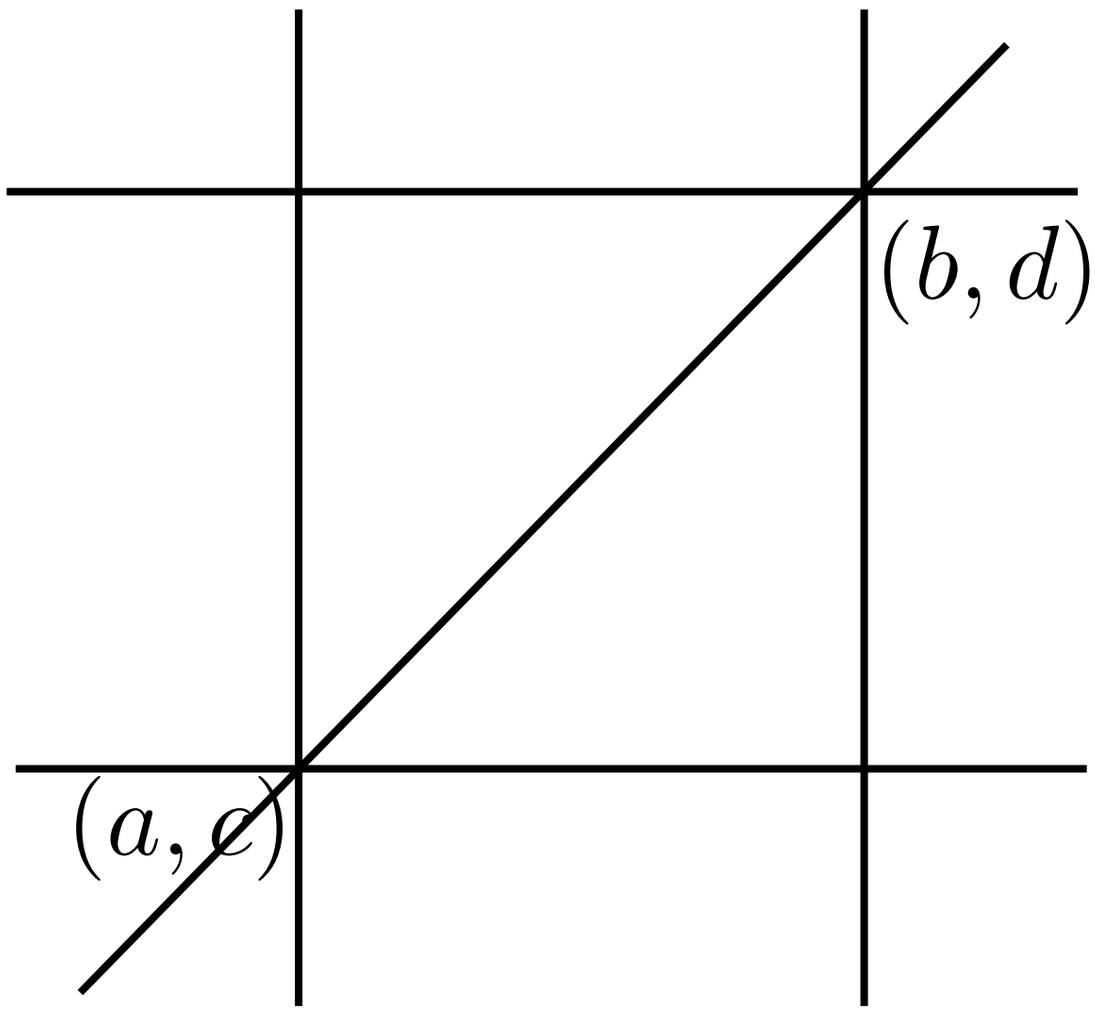


Hybridization - Linear Approximation

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

Find a linear function which interpolates $f(x)$ at the vertices



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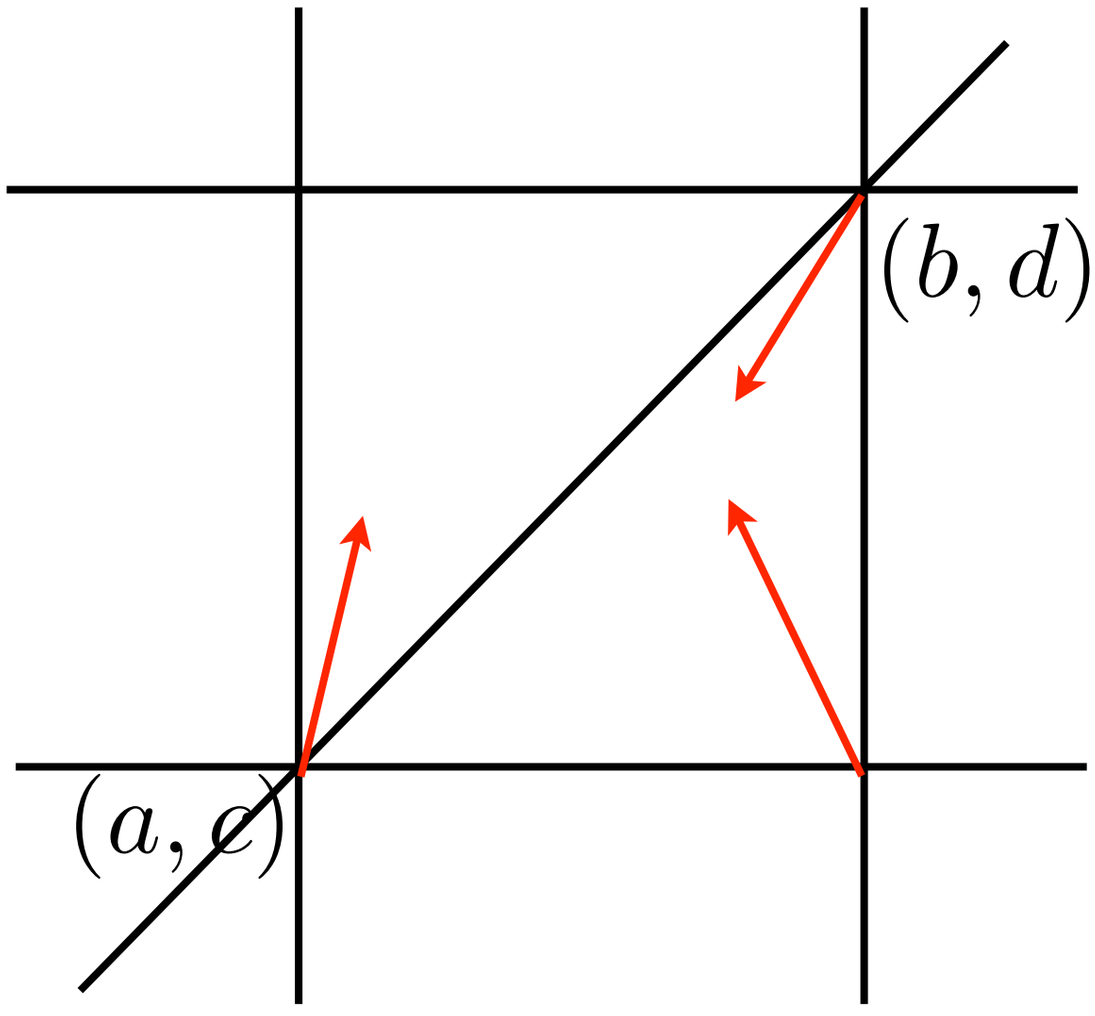

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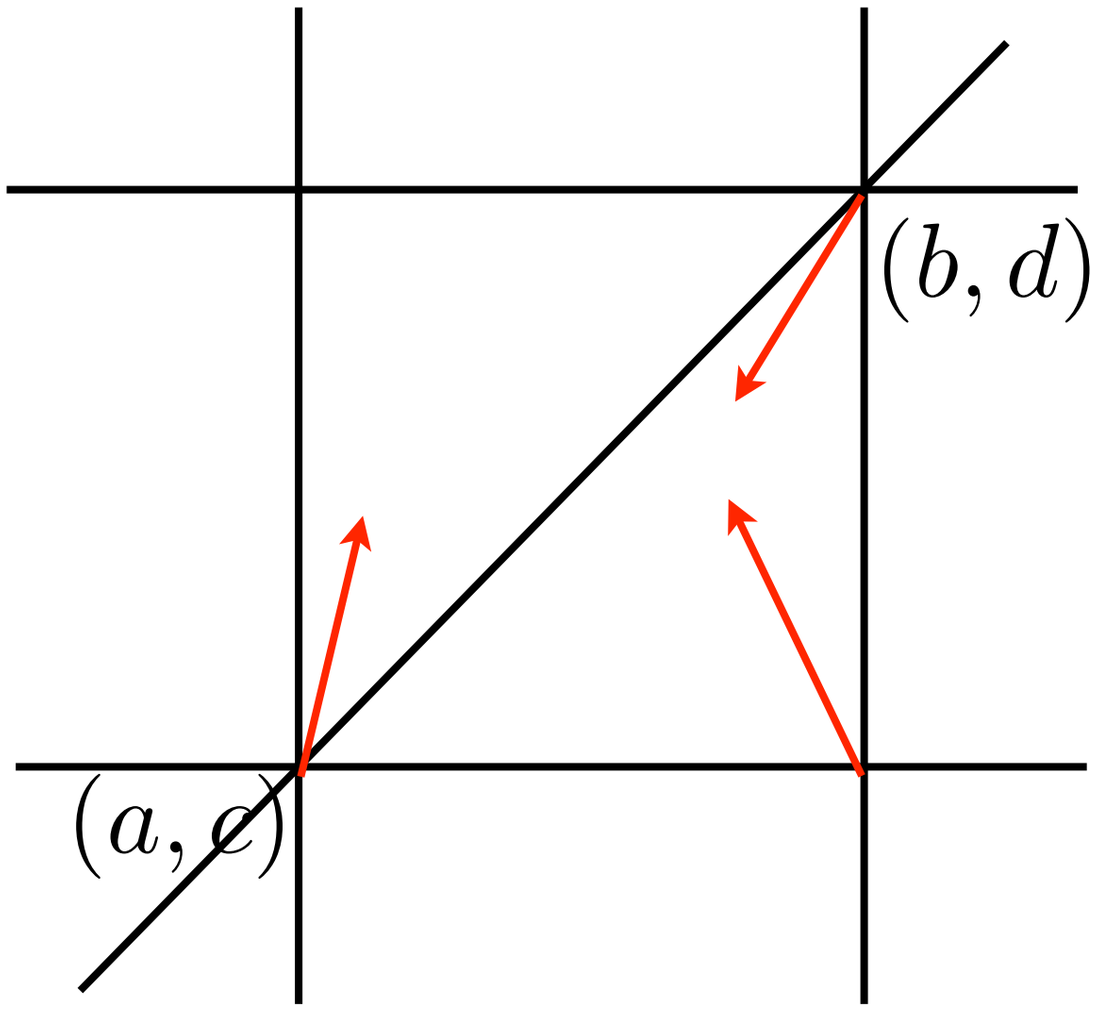

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Bounded error approximation in a finite time interval by choosing small enough cells, for Lipschitz continuous functions

What about approximations for infinite time?

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- Finite time
  - Bounded error approximation
  - Construct finer abstraction by reducing the error bound

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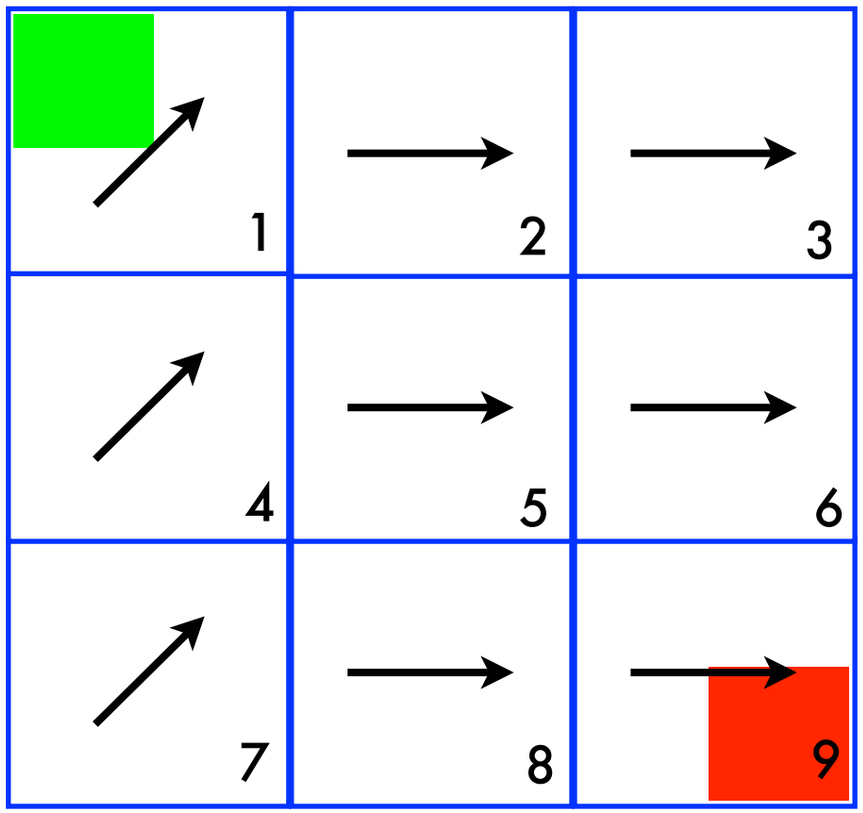
  - In general, no bound on the error
  - Refine based on a counter-example

Abstraction



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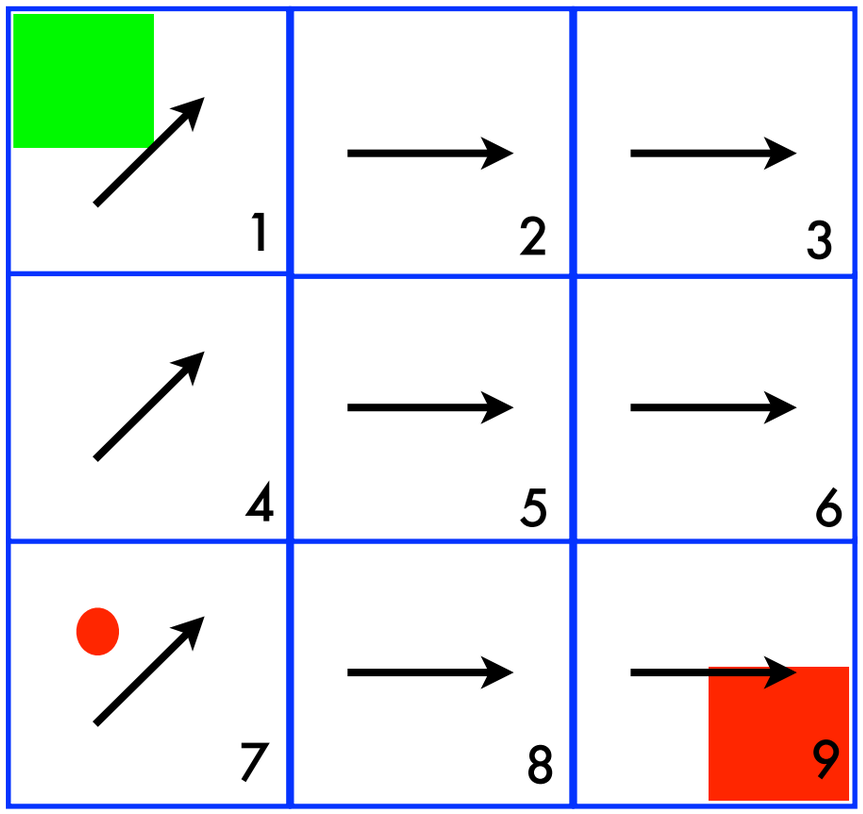

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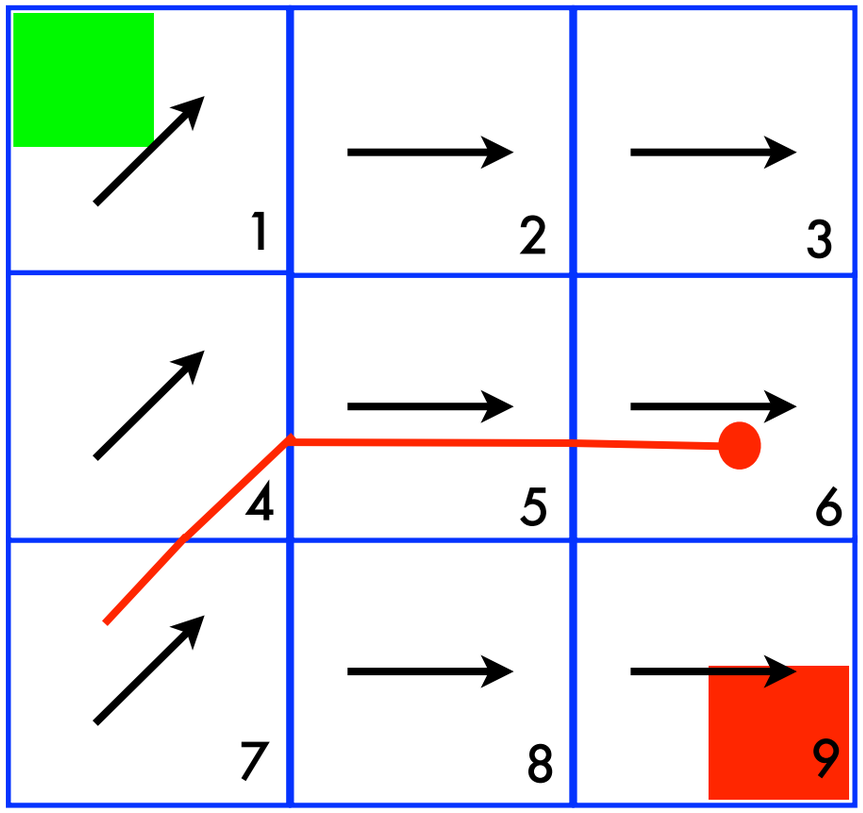

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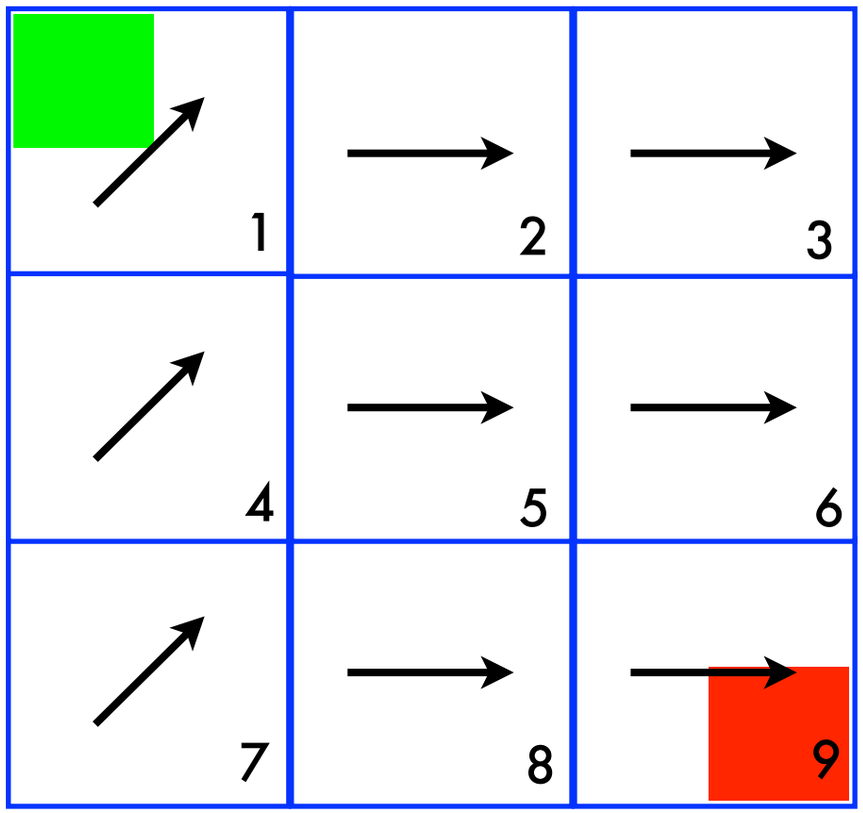

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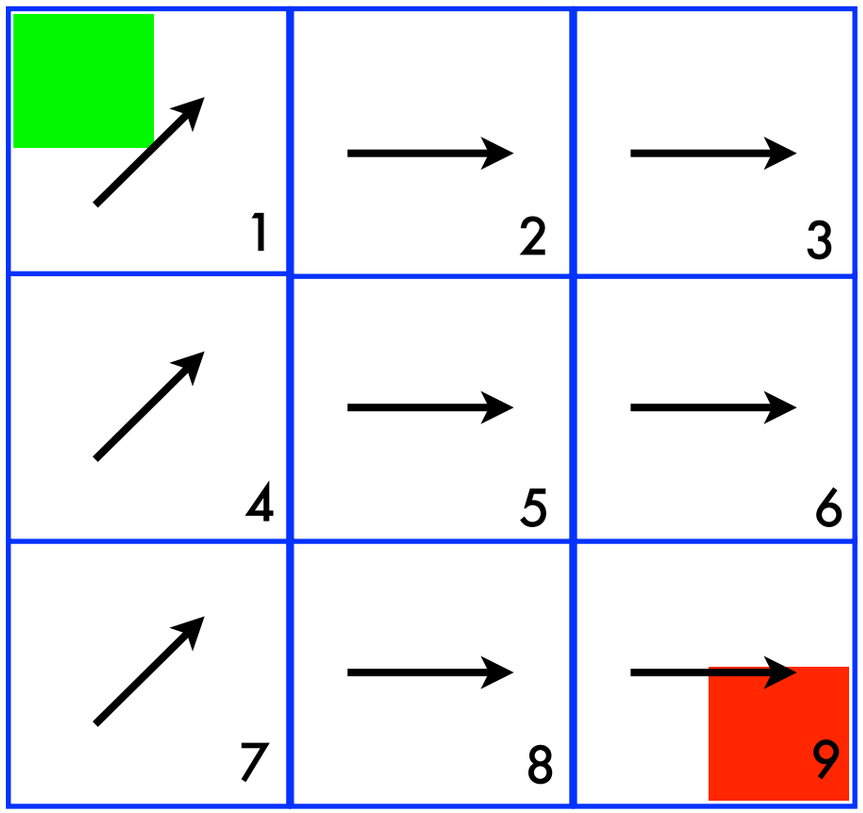

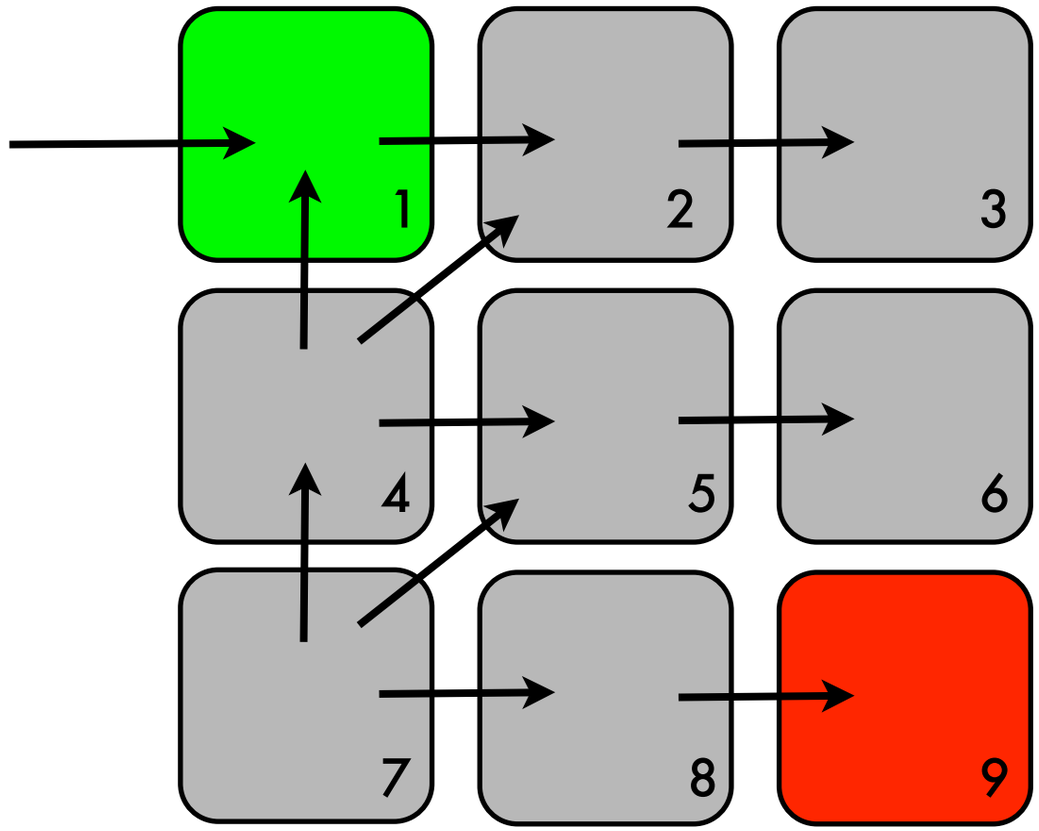

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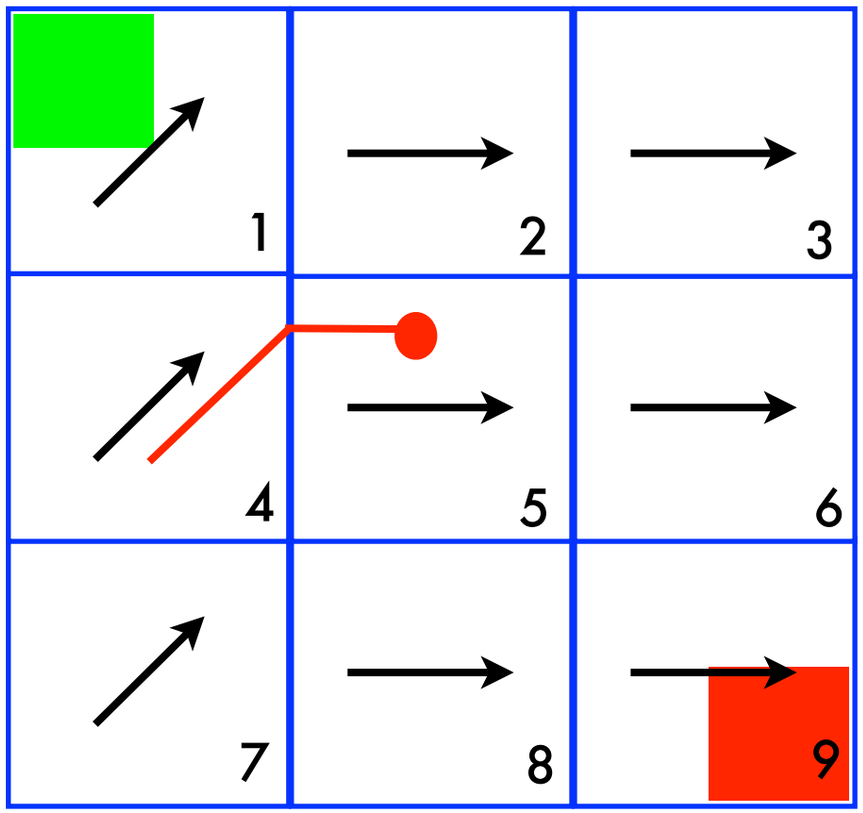

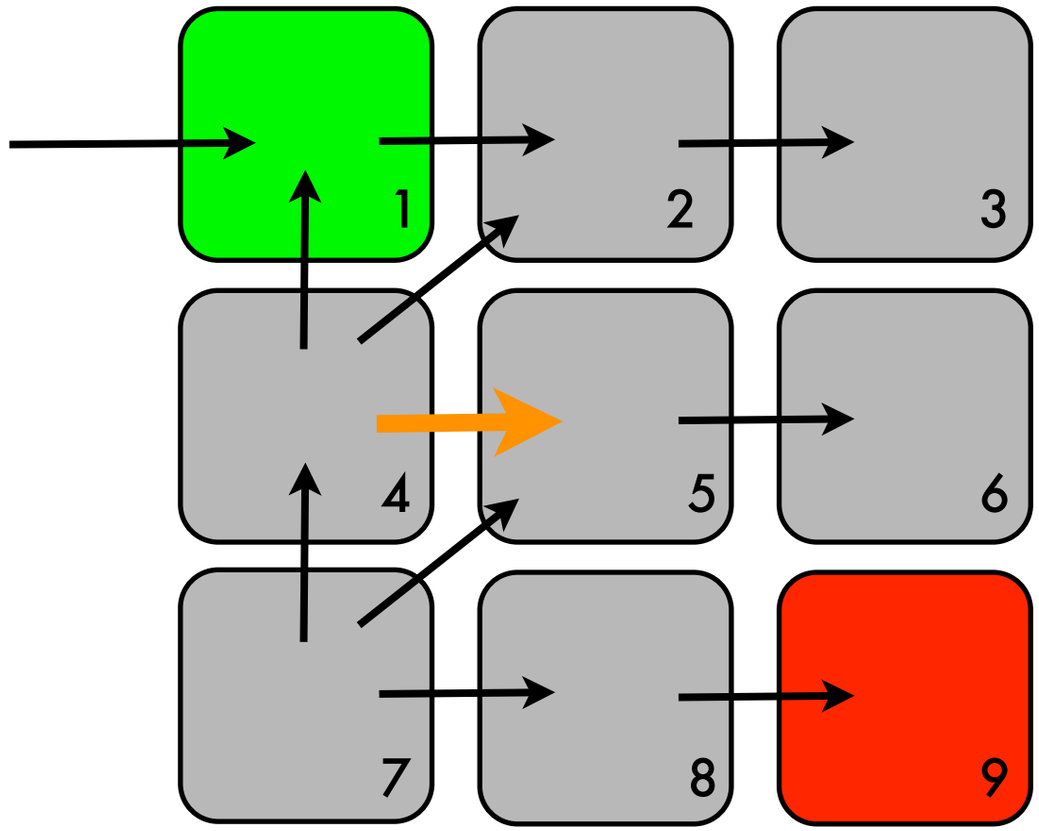

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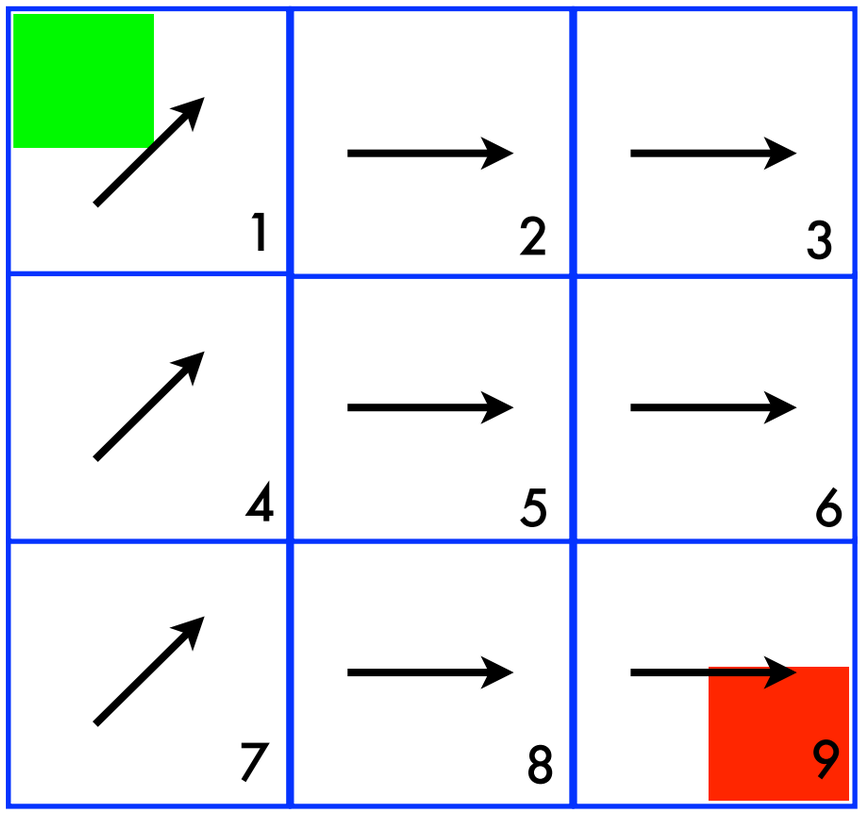

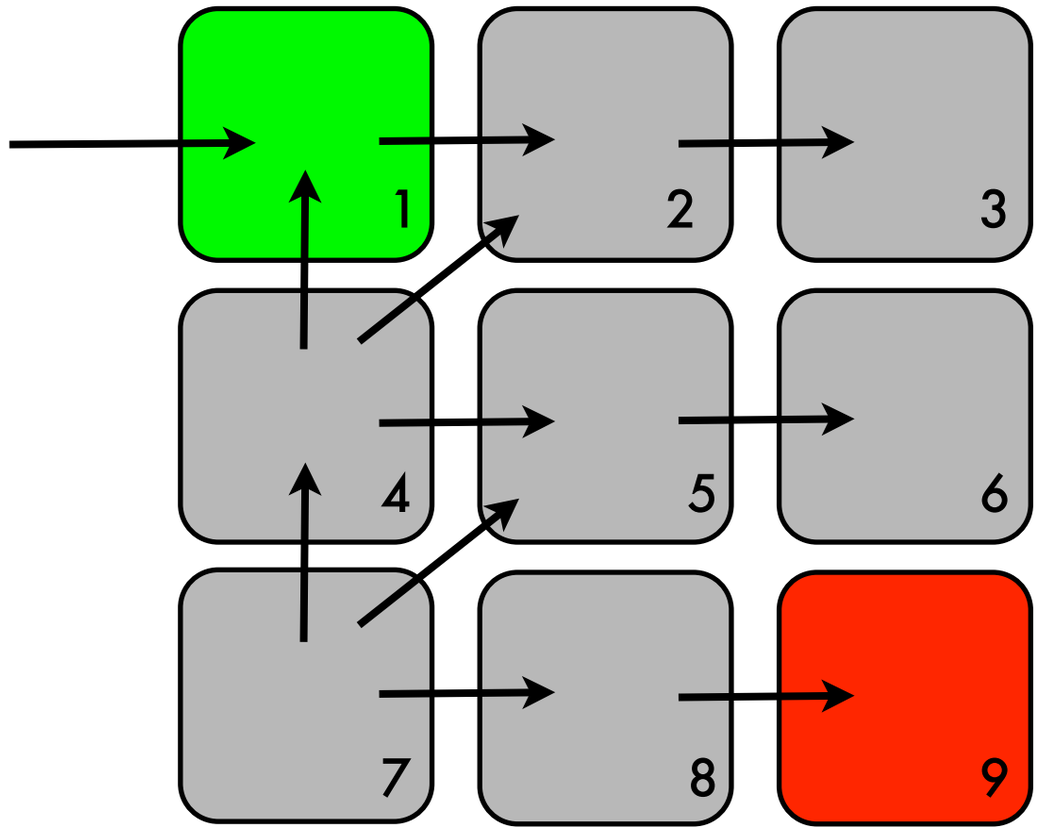

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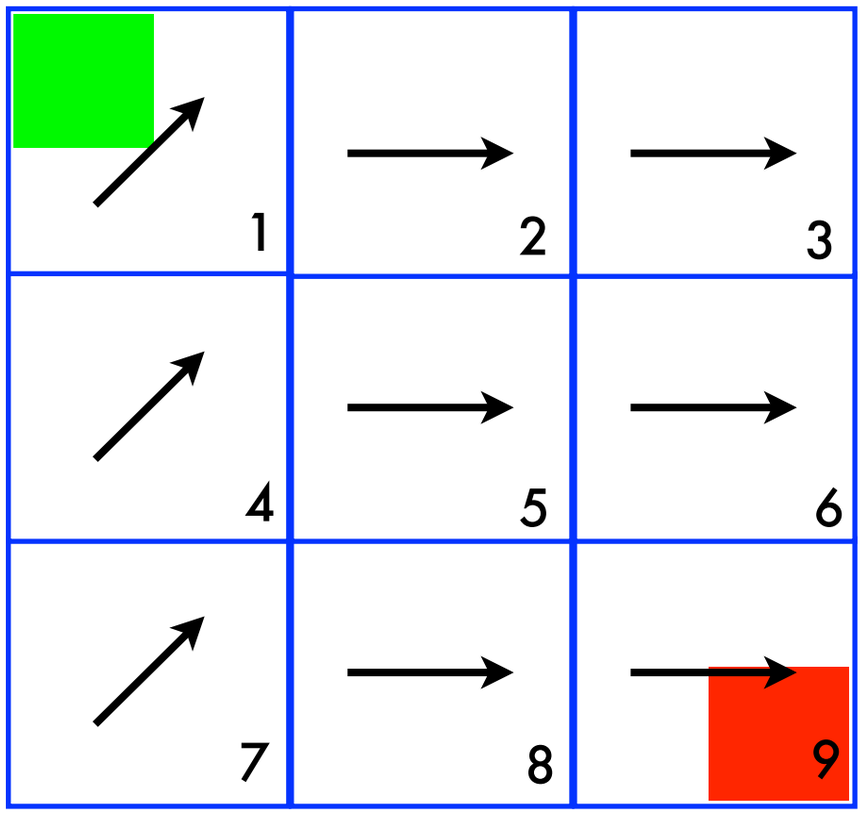
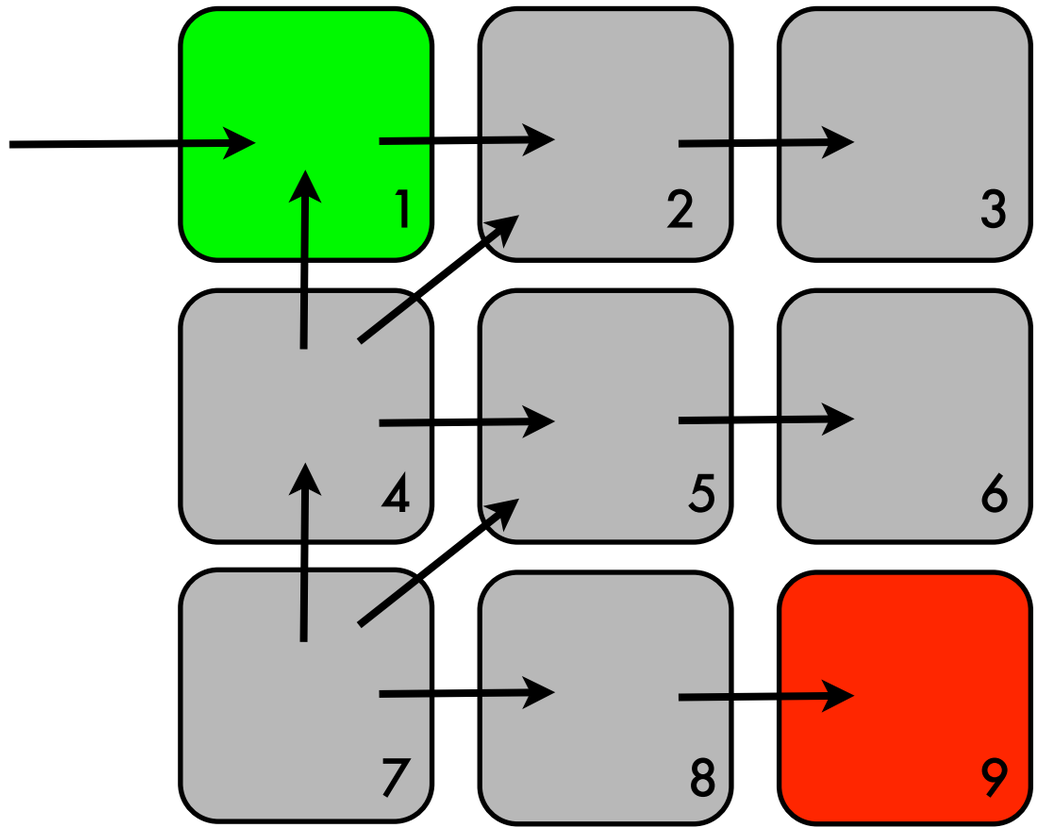
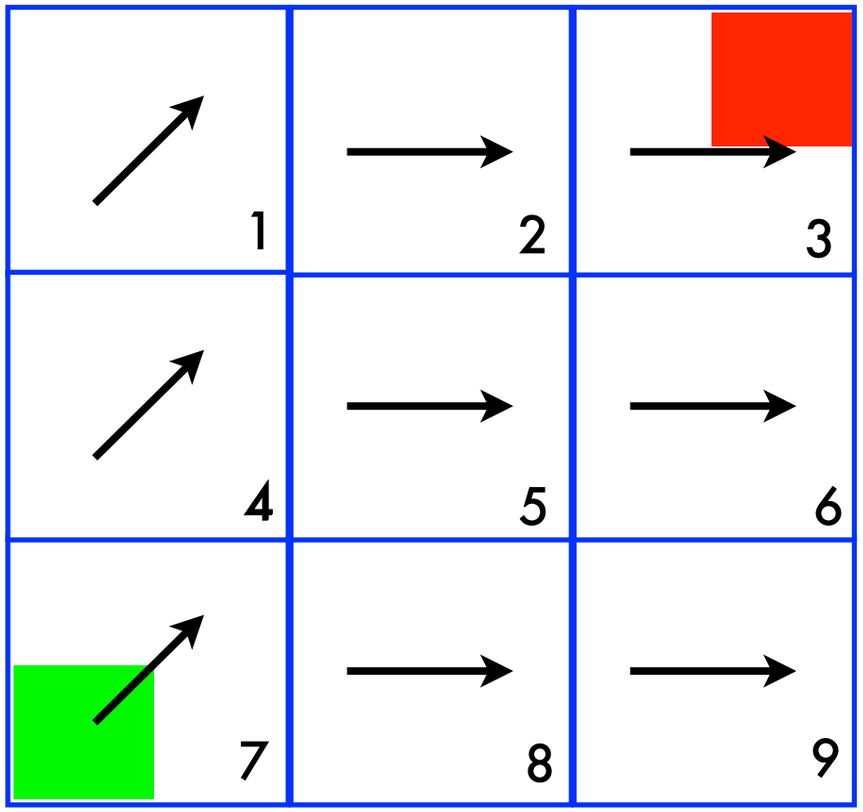
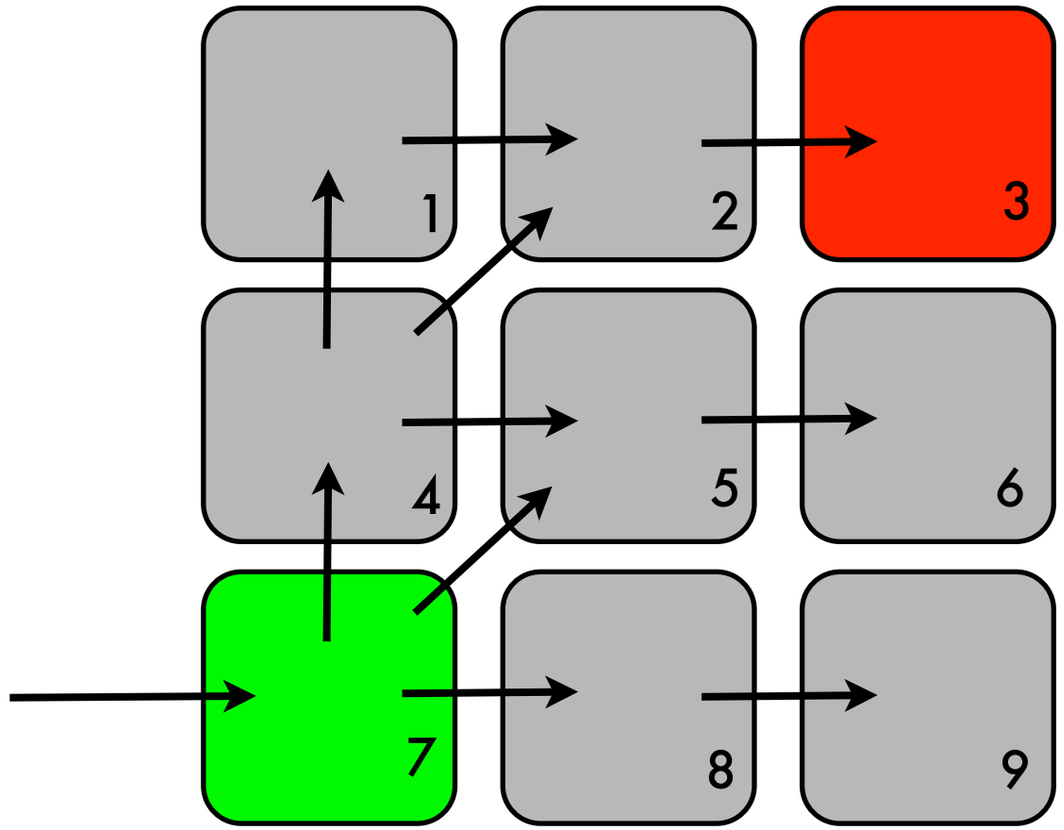




For every trajectory of the robot, there is a corresponding path in the abstract graph

Abstraction





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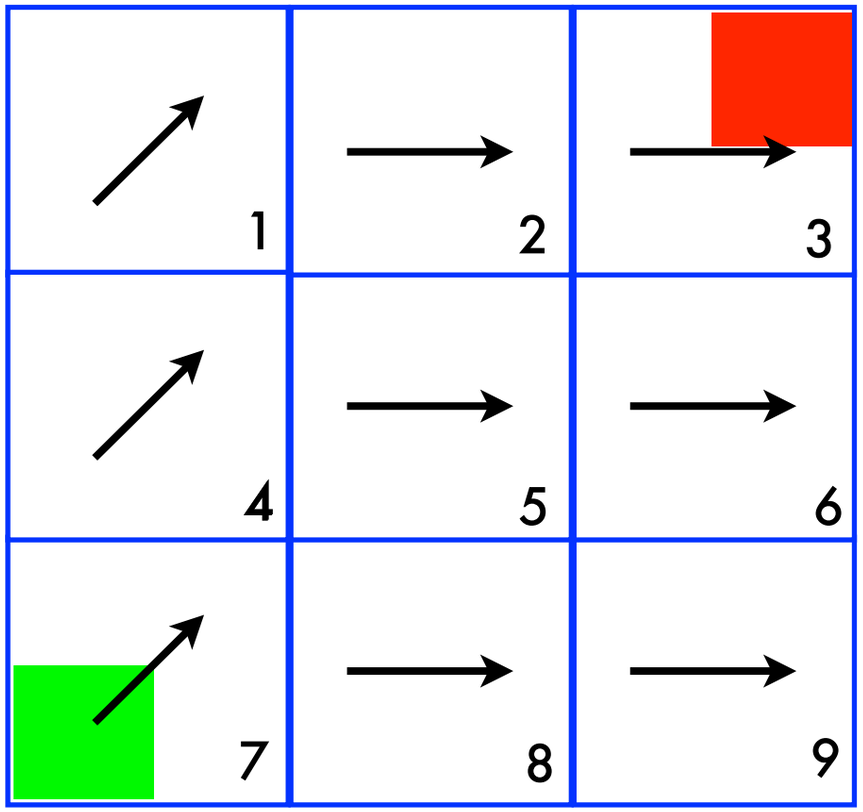

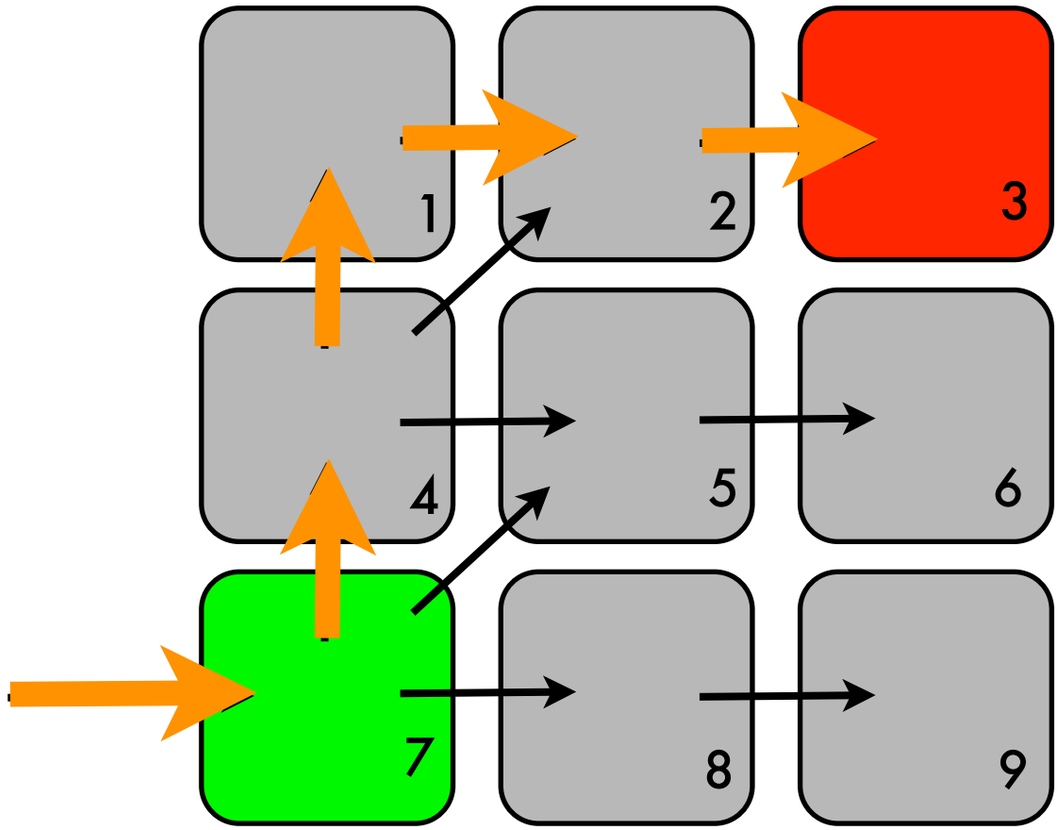

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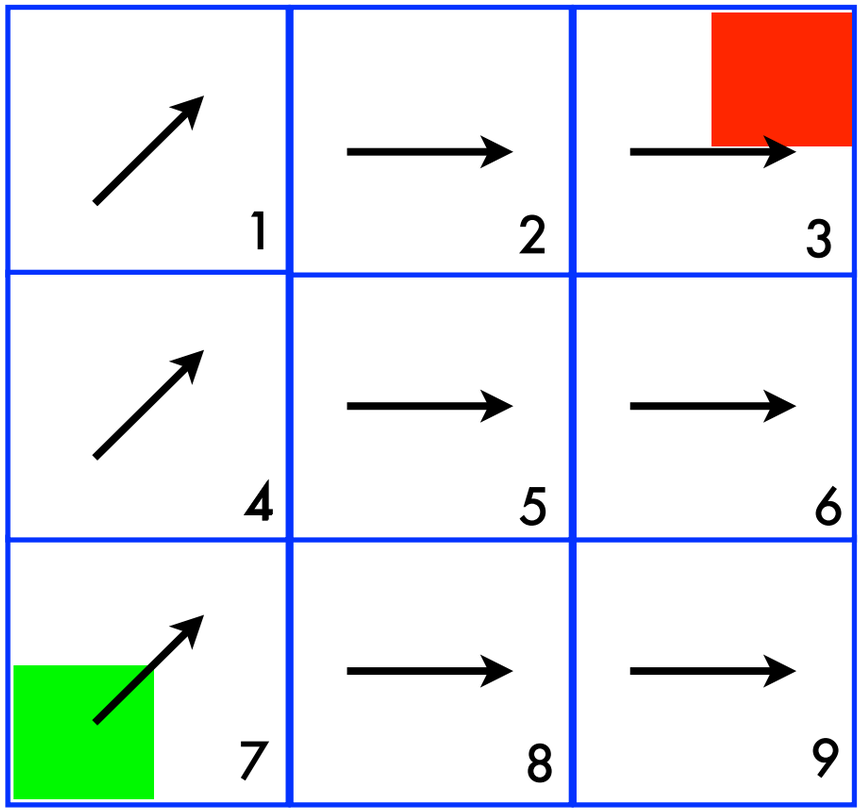
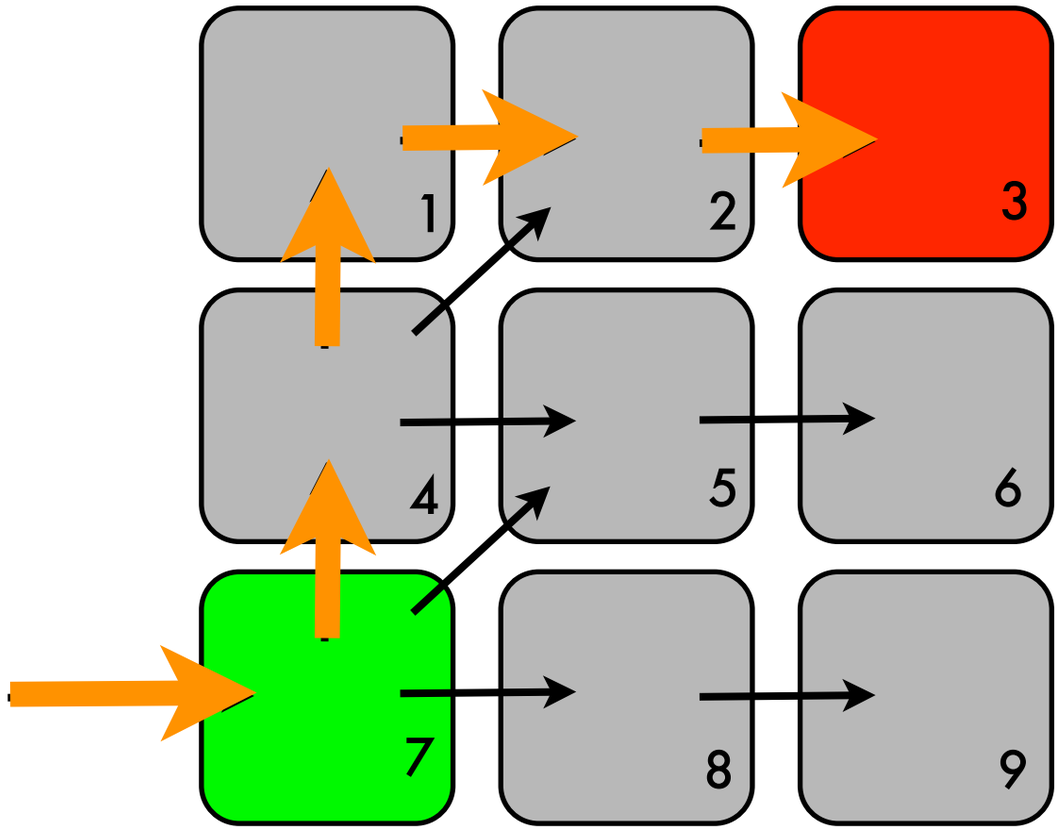
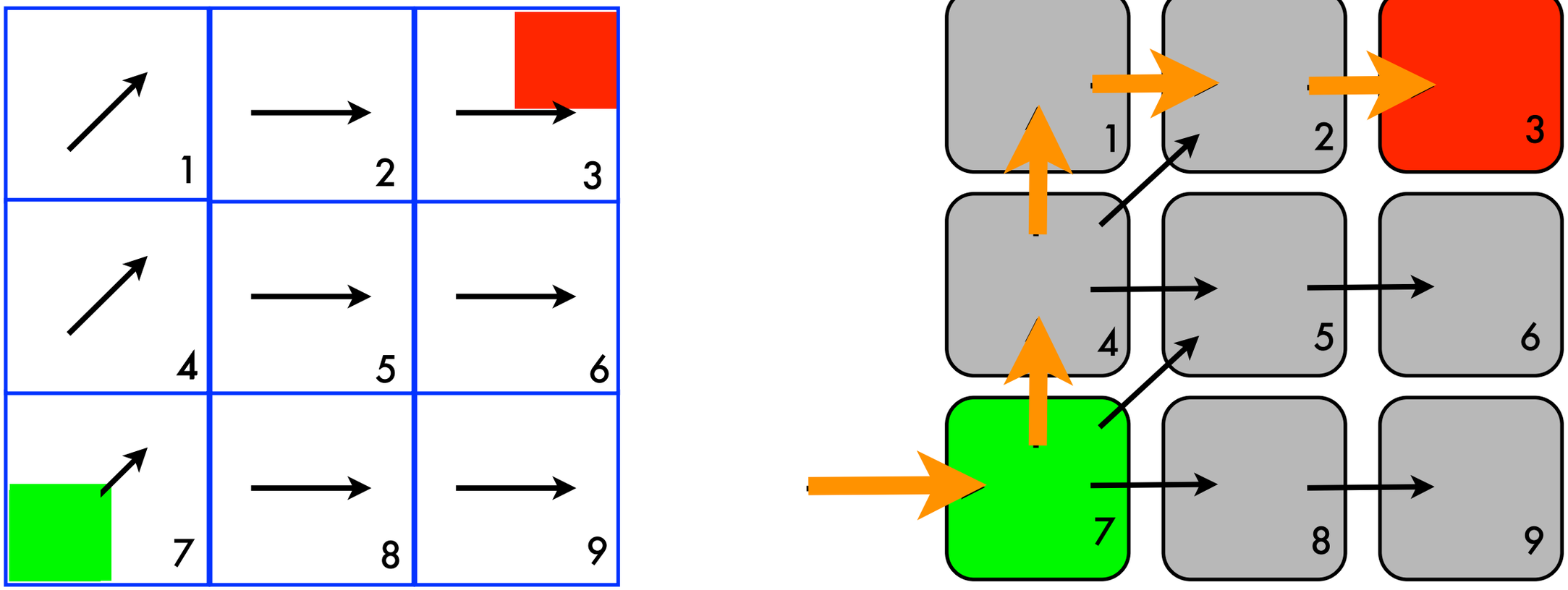




Right abstractions hard to find!

Refinement



Refine the abstraction

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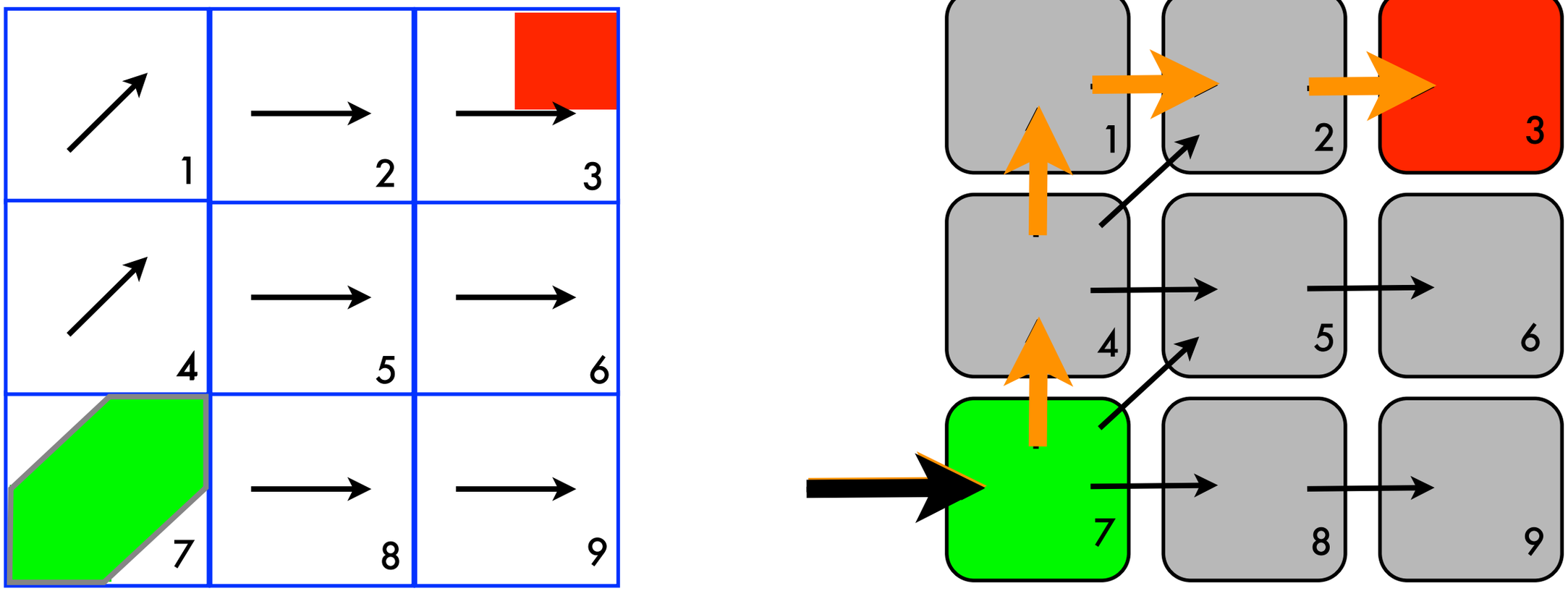

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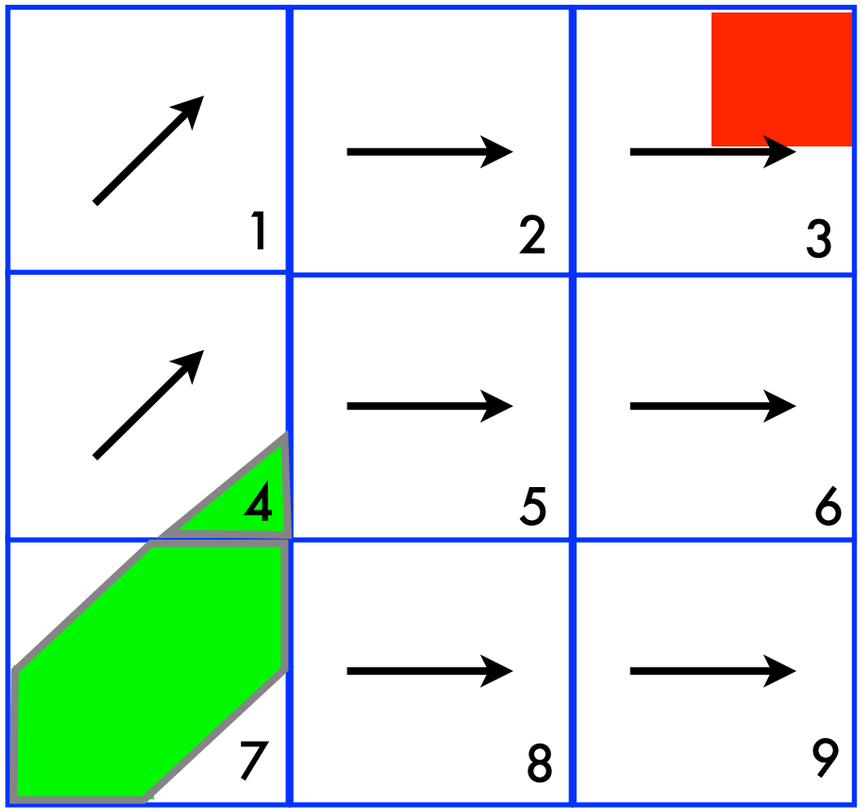

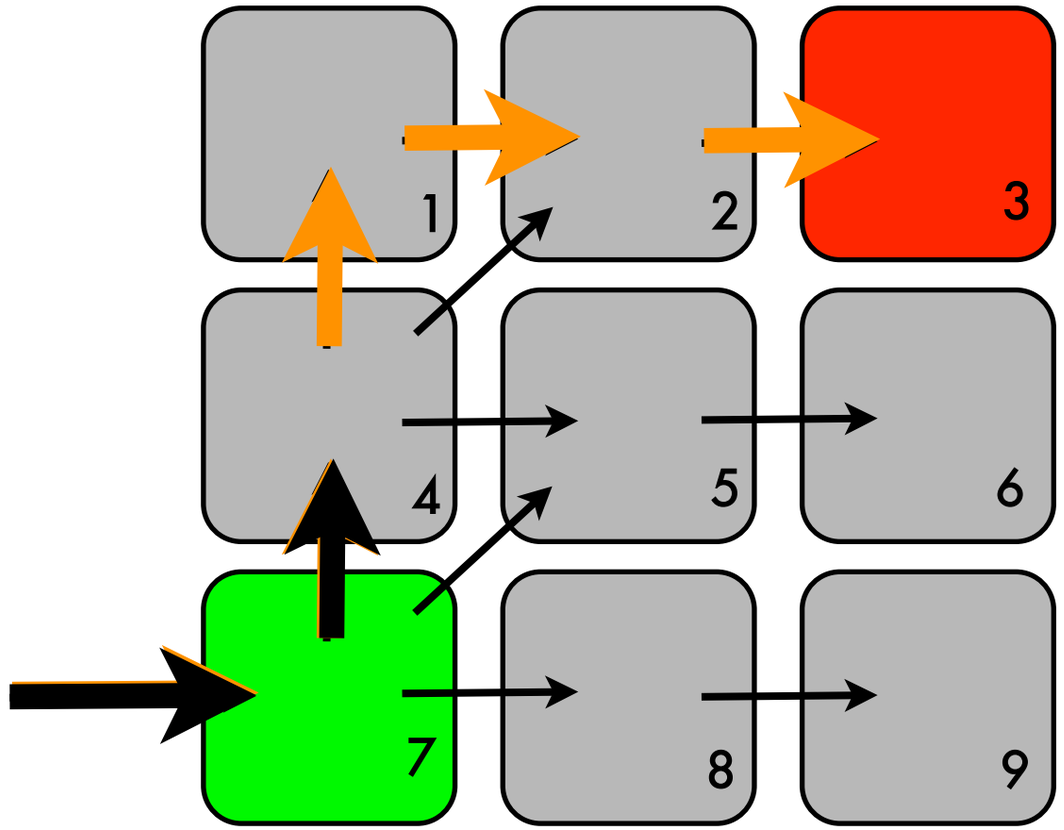

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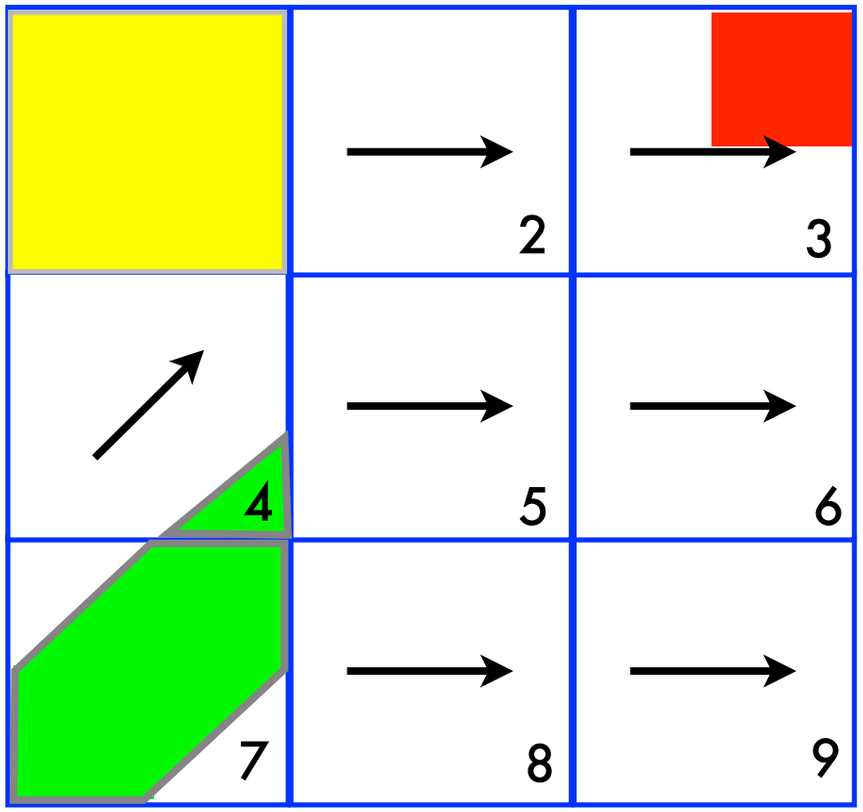

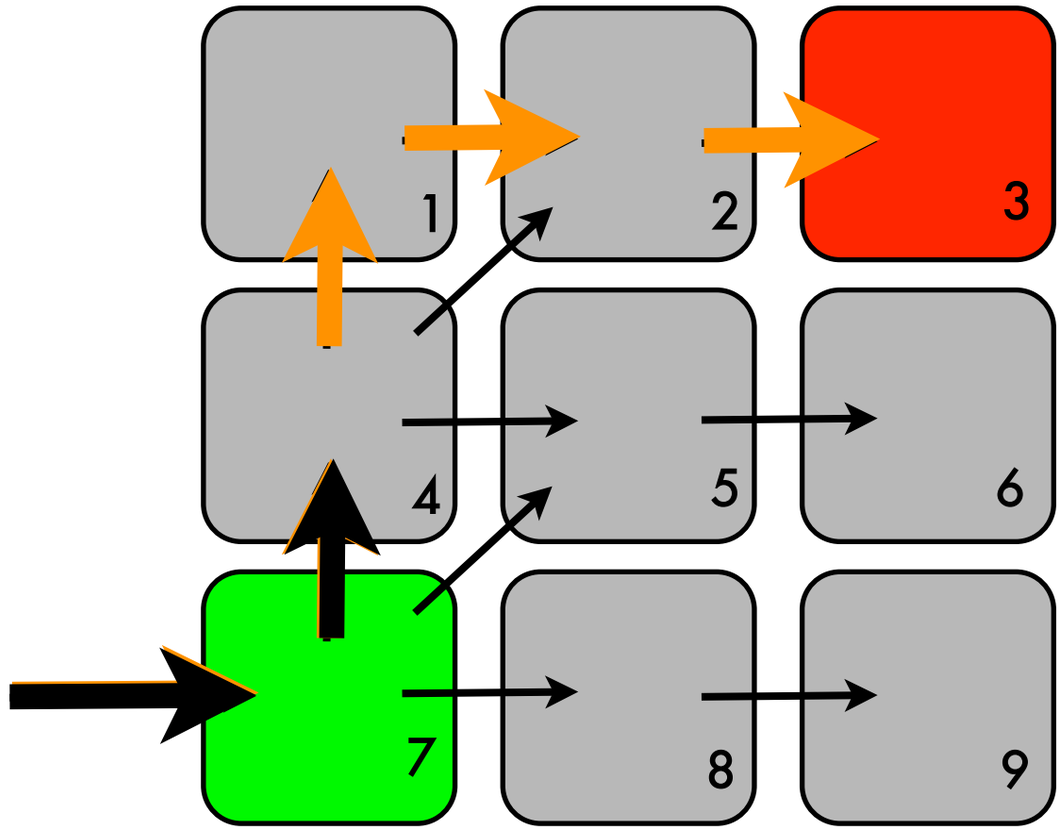

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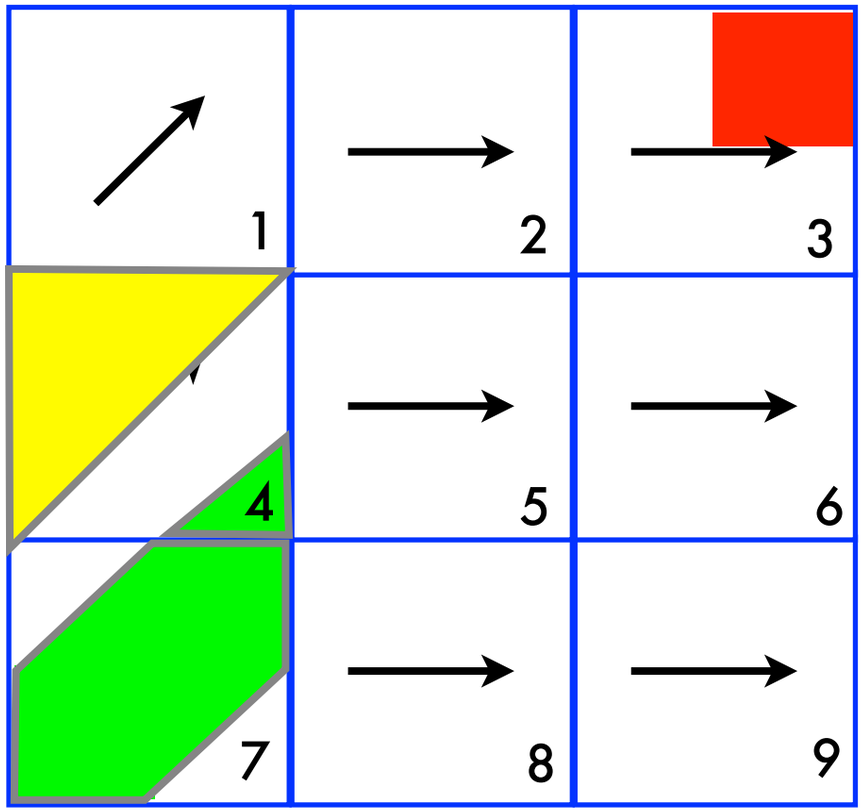

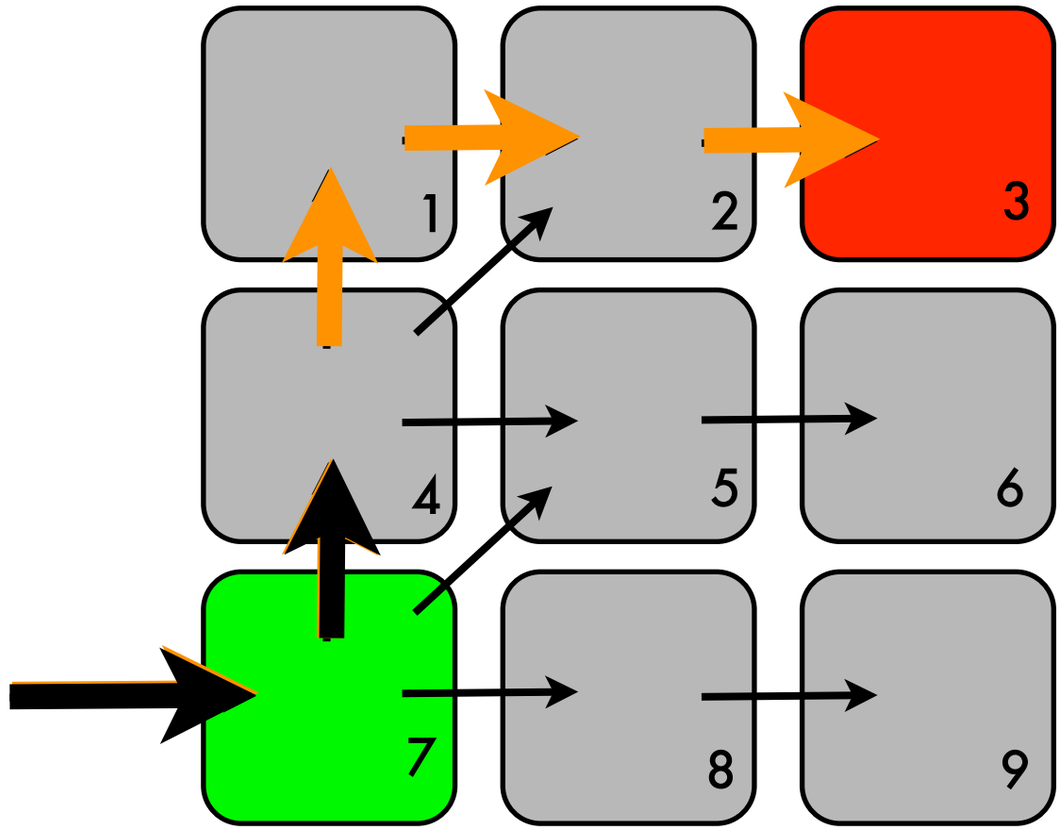

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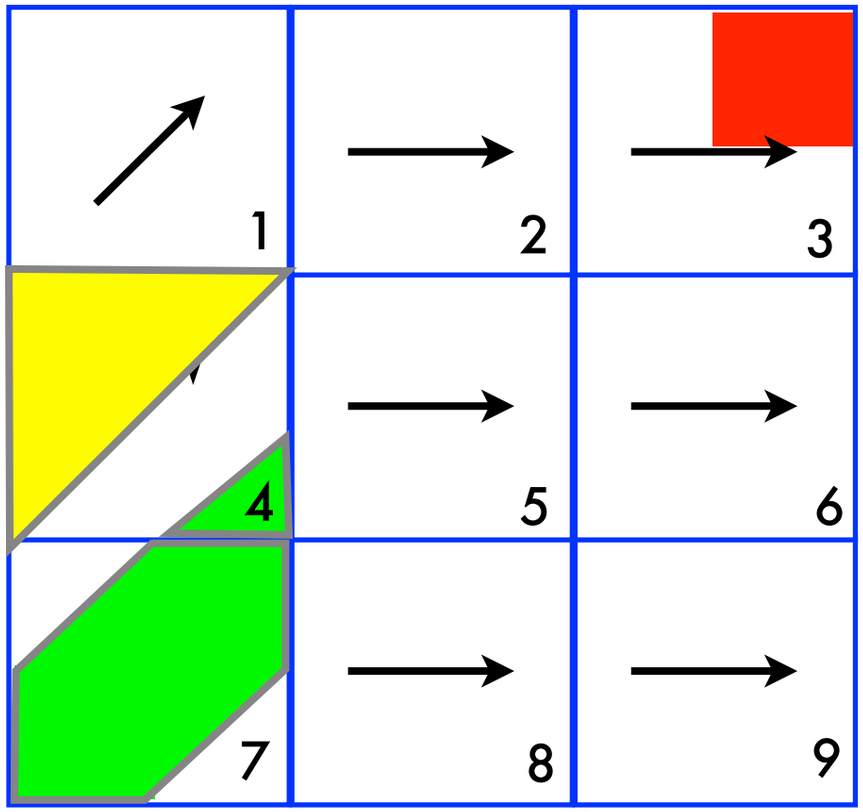
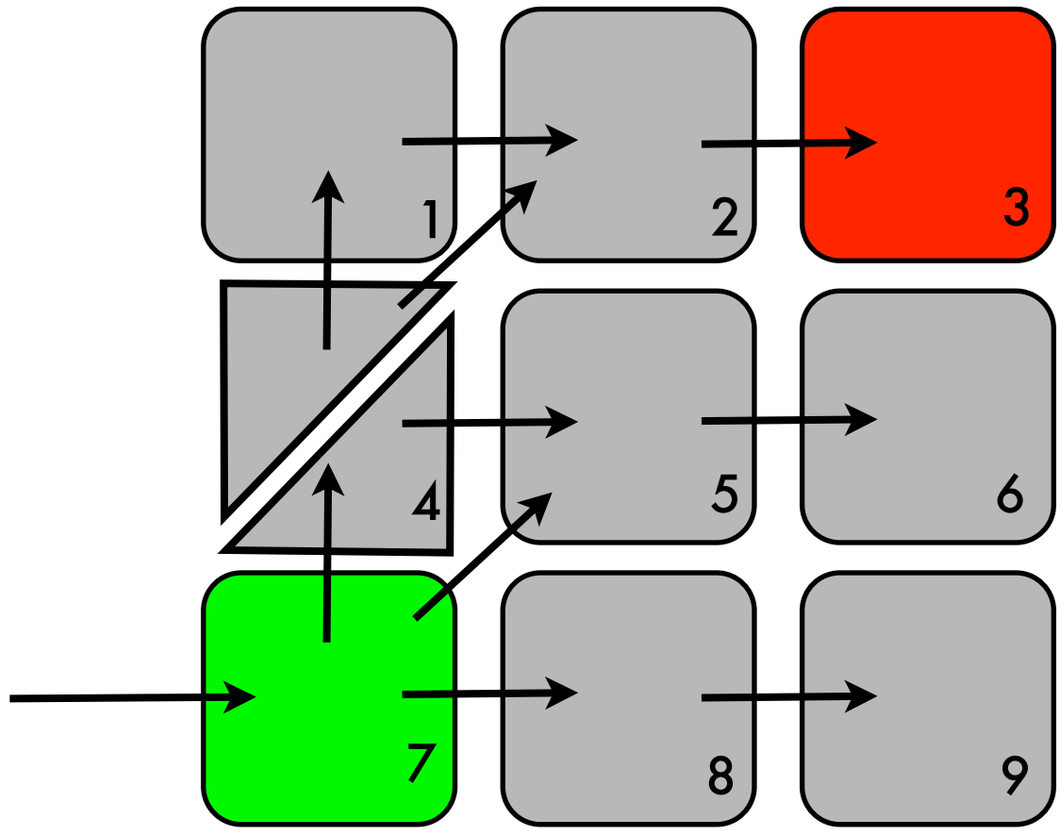
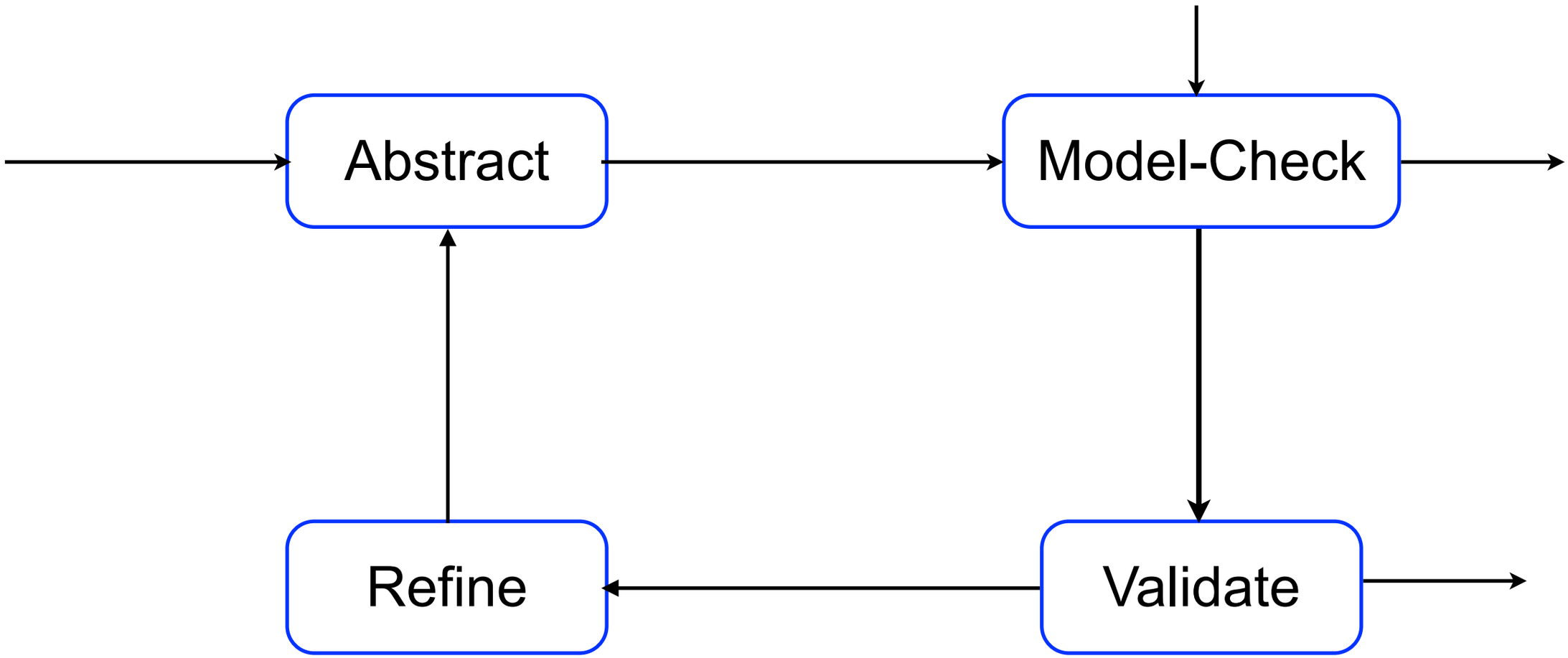




Refine the abstraction

Counter-Example Guided Abstraction Refinement



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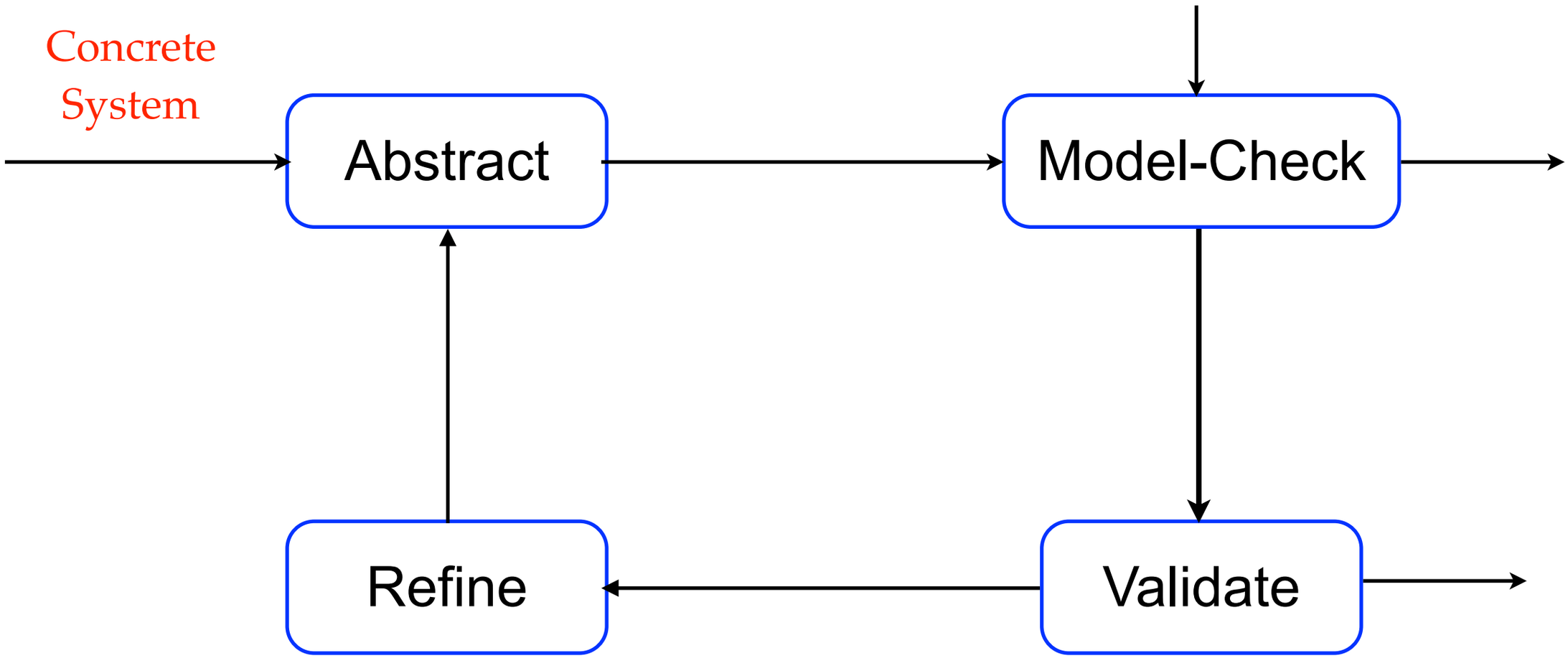

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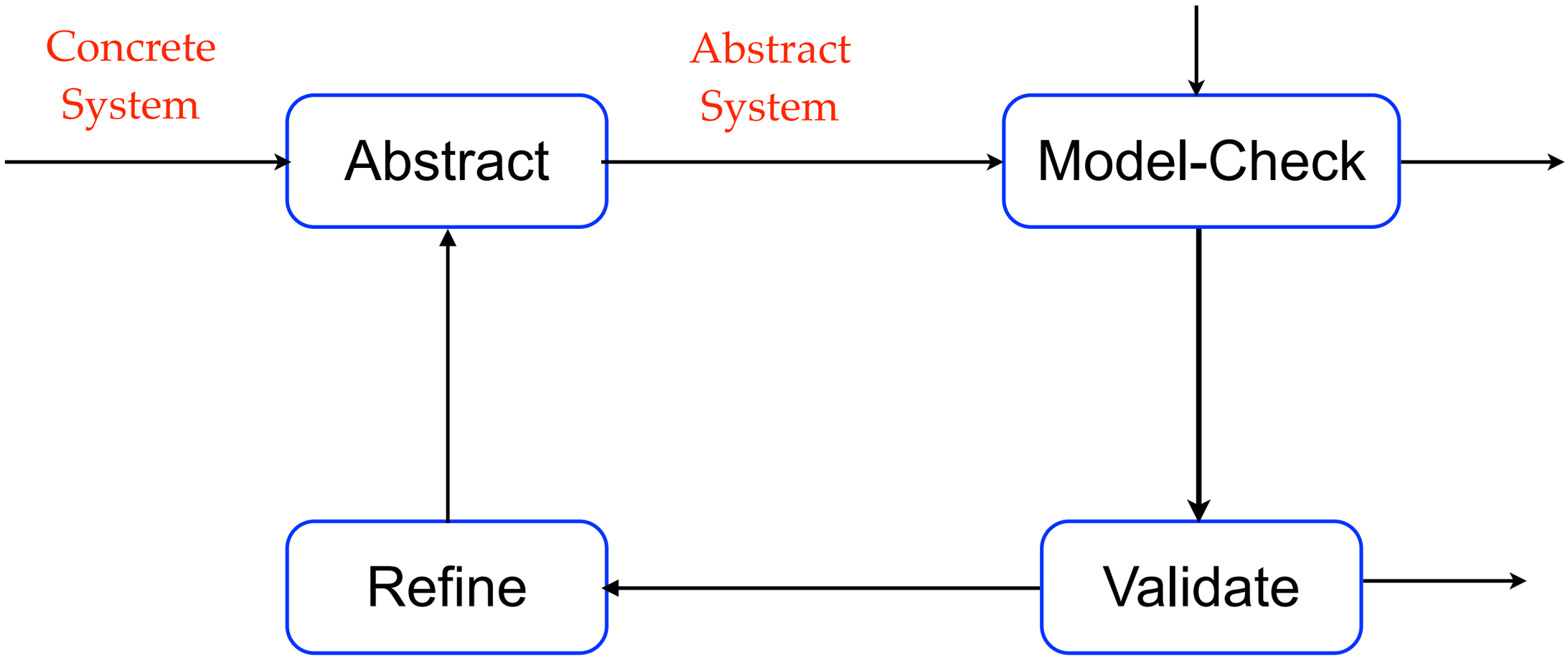

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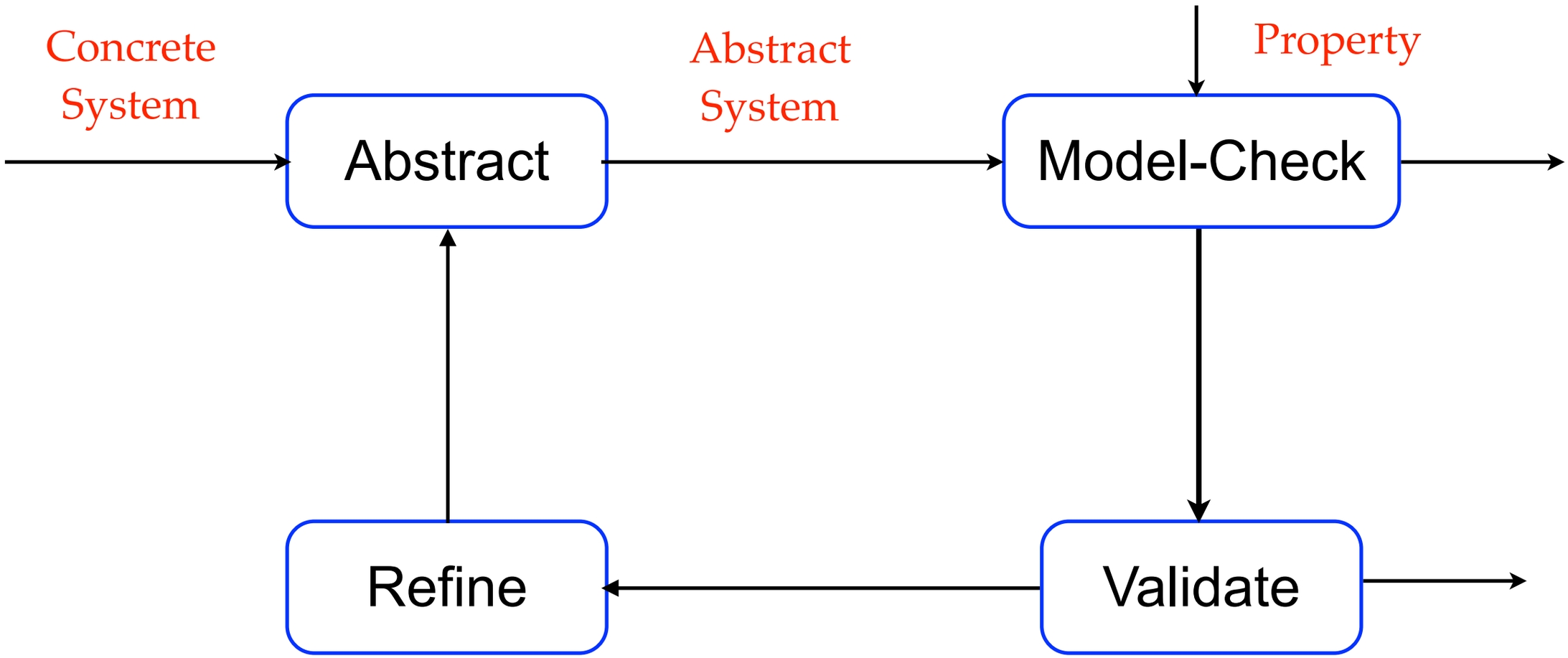

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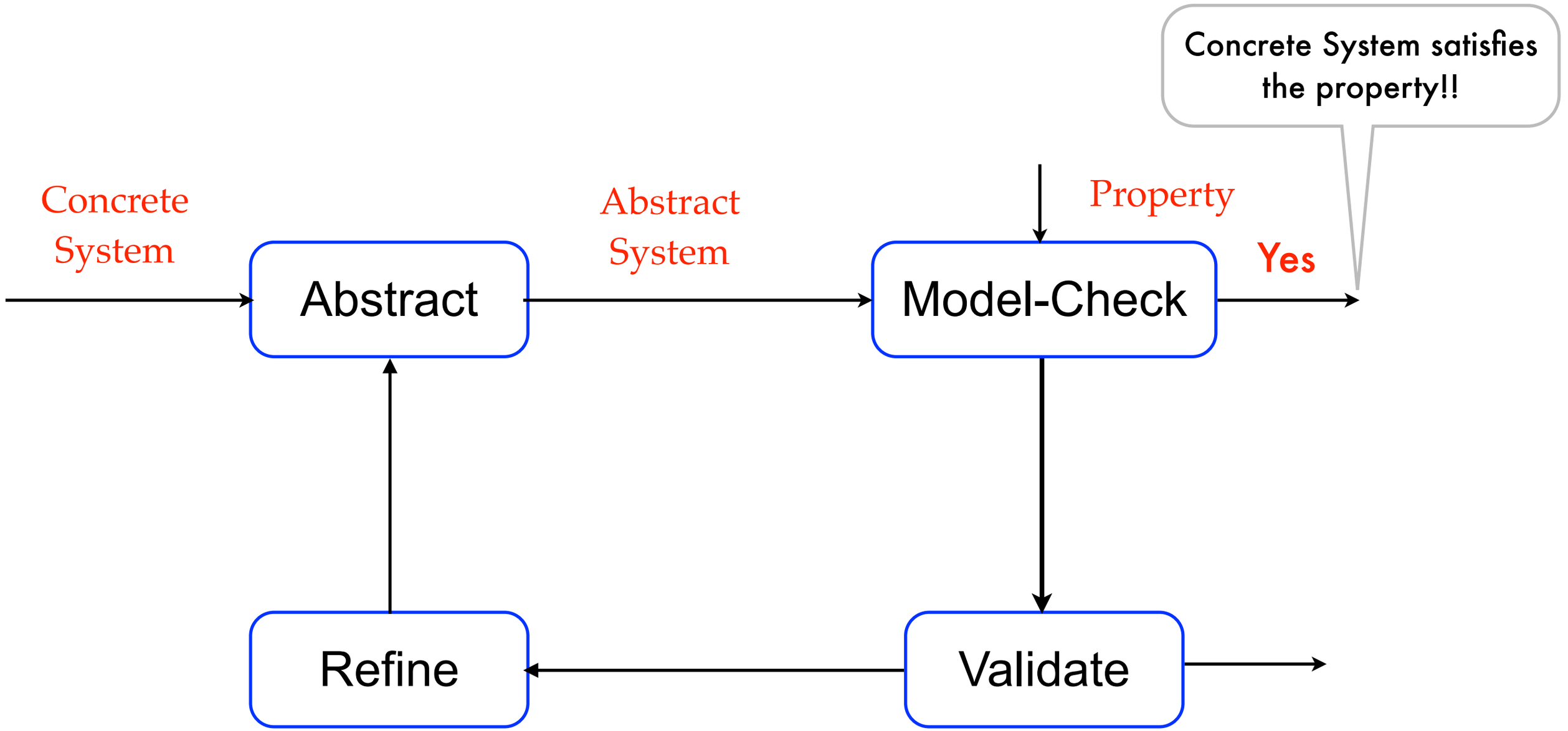

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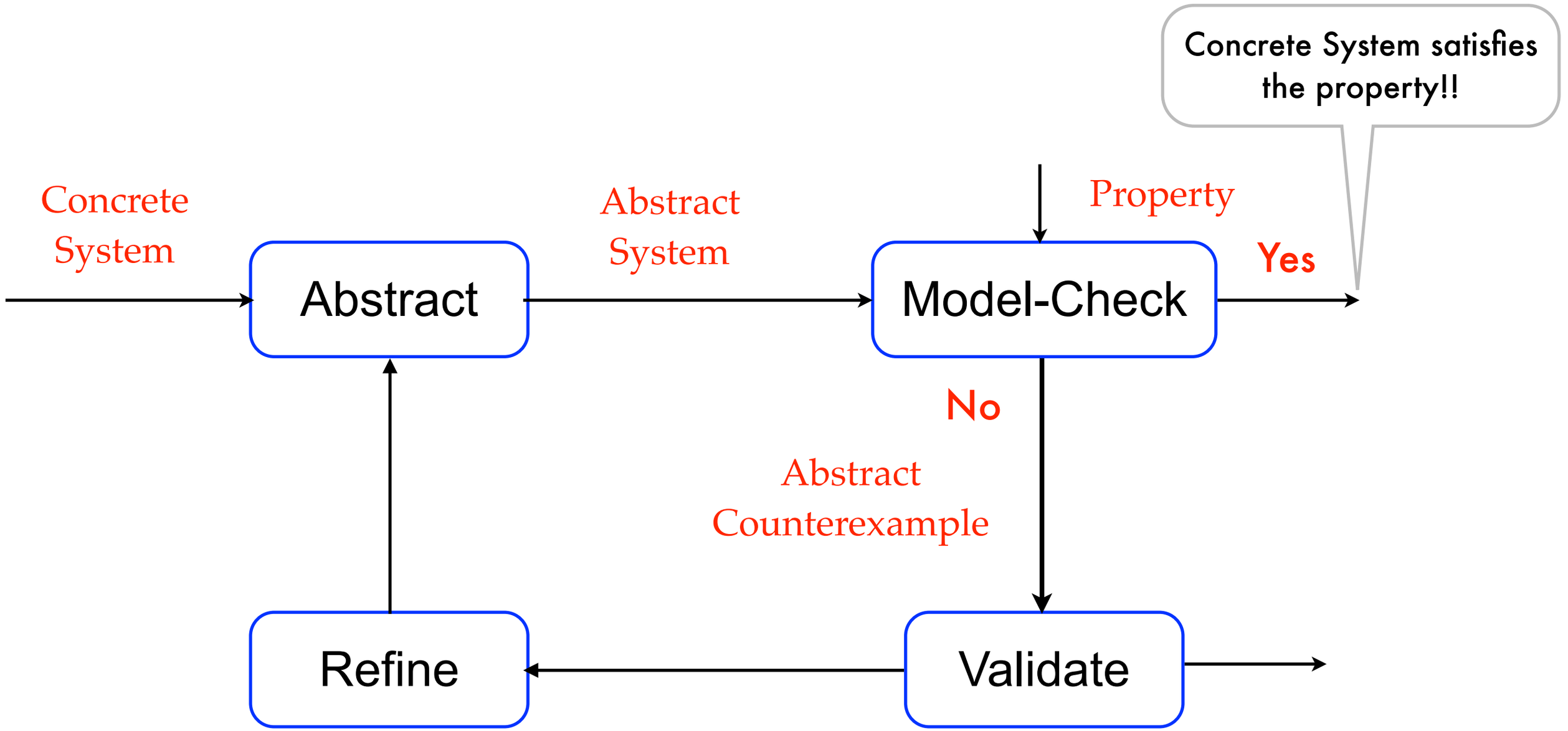

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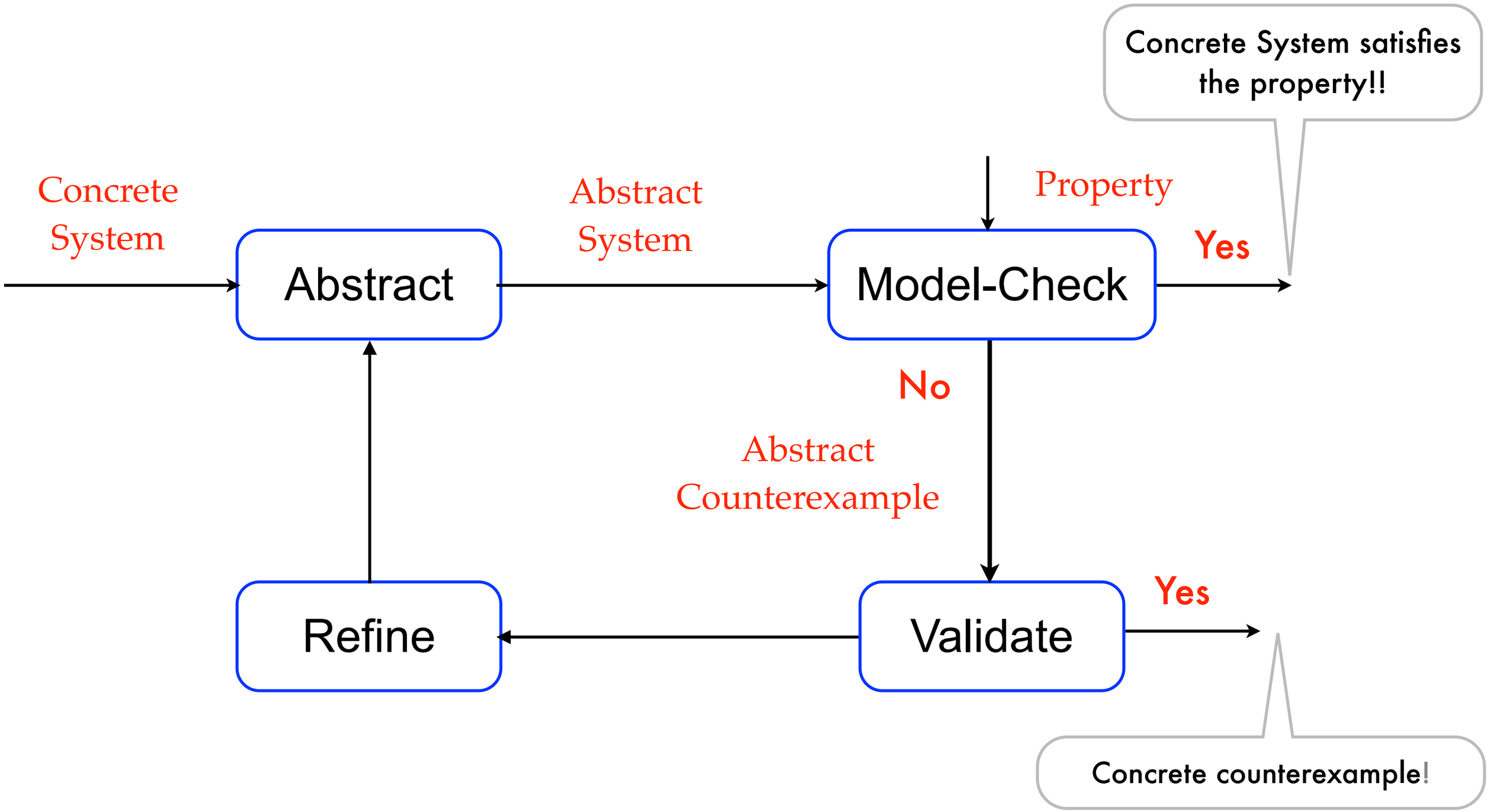

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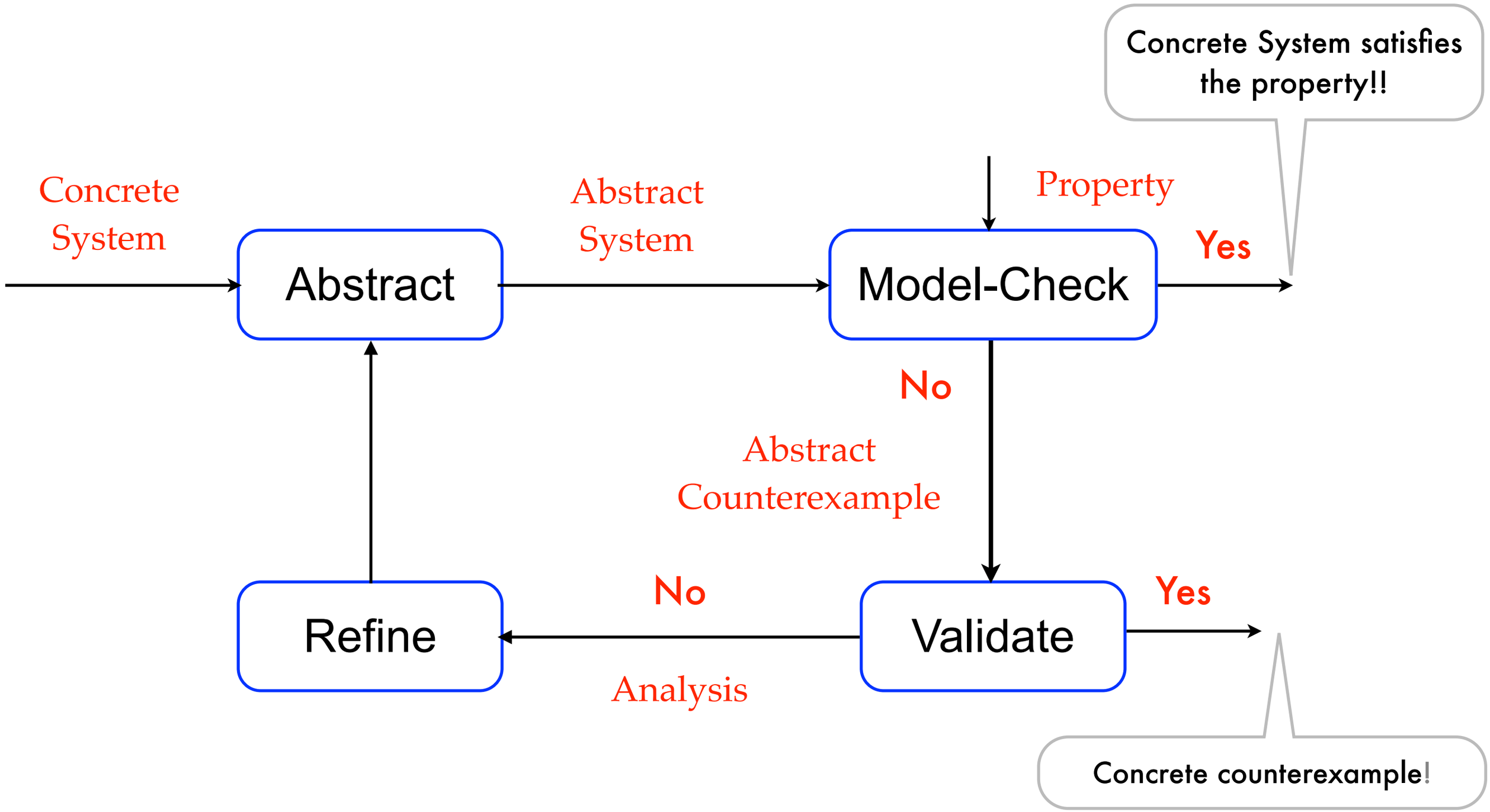

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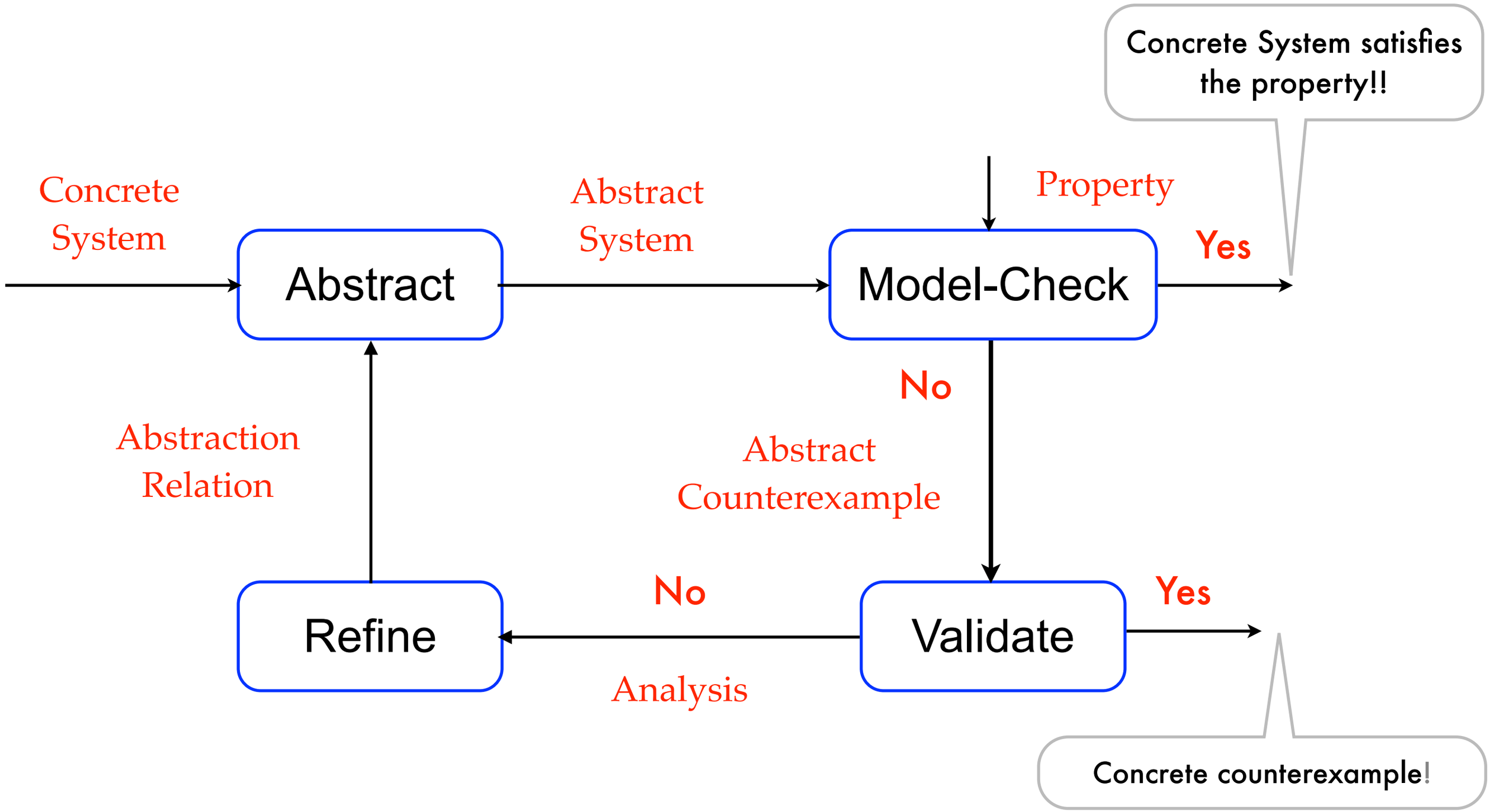
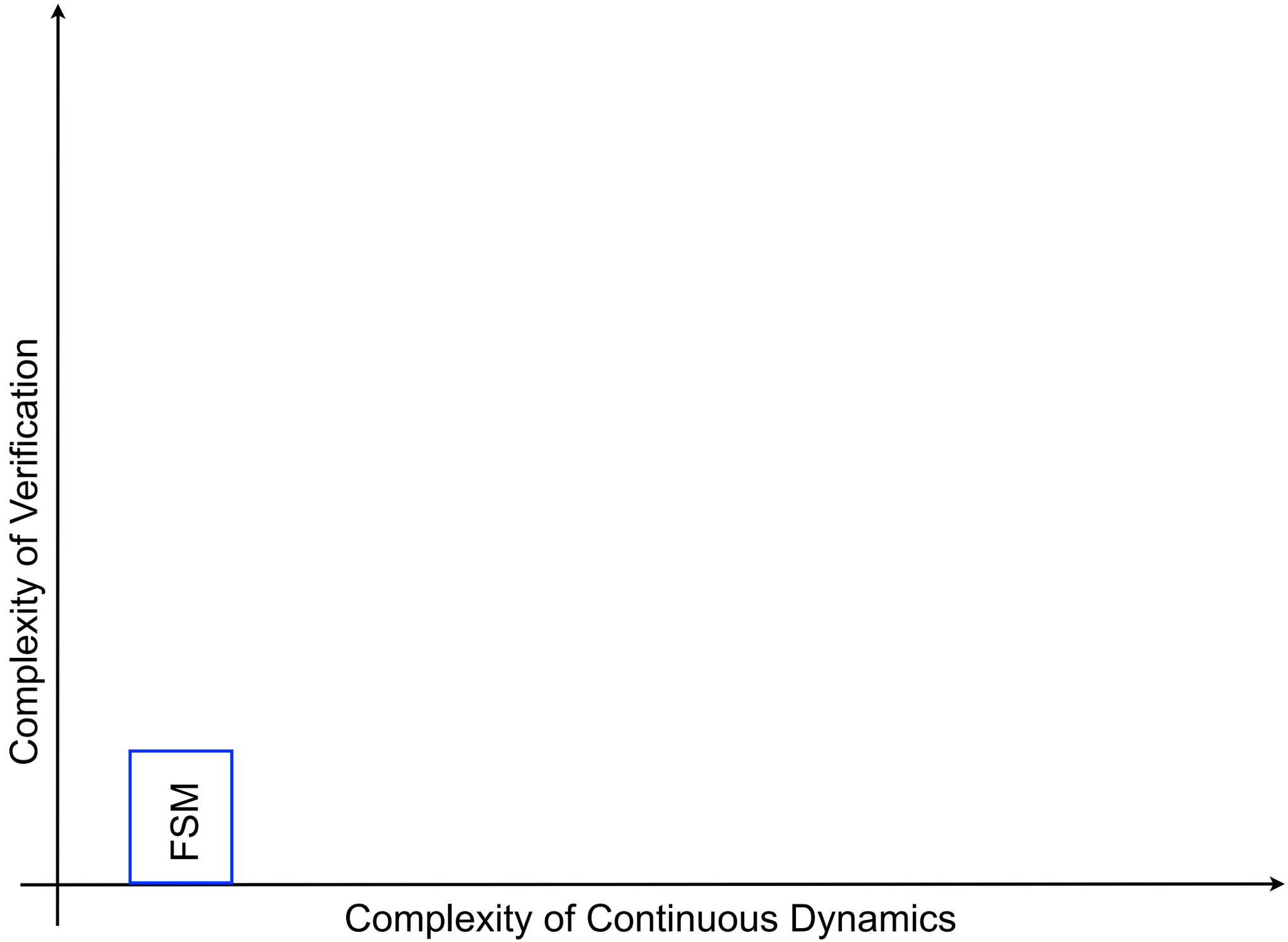


Related Work

- Software Verification [Kurshan et al. 93], [Clarke et al. 00], [Ball et al. 02]
  - SLAM, BLAST
- Discrete CEGAR for hybrid systems [Alur et al. 03], [Clarke et al. 03]
- Hybrid CEGAR for hybrid systems [P.,Duggirala,Mitra,Viswanathan], [Dierks, Kupferschmid, Larsen])

Summary of safety verification



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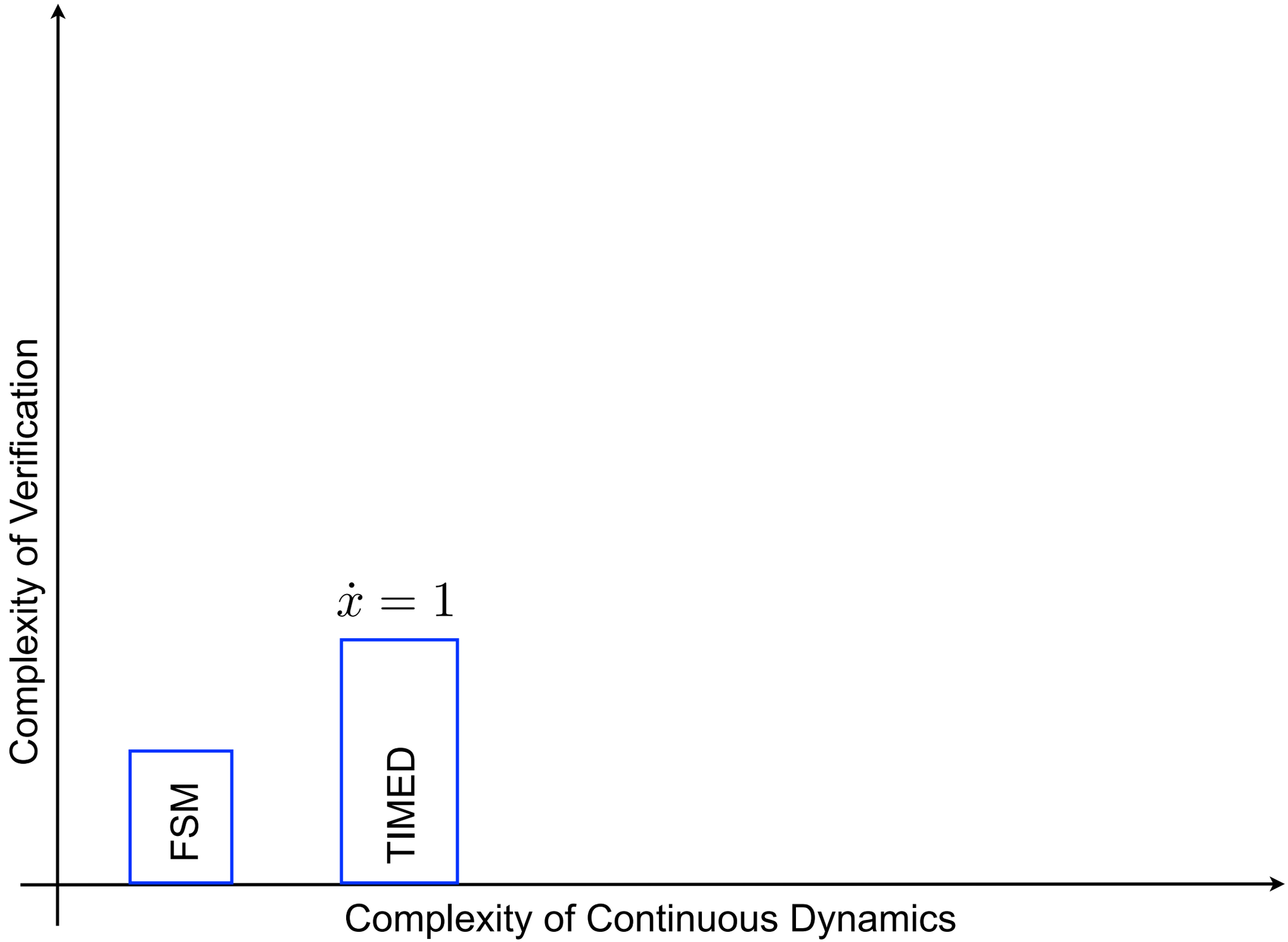

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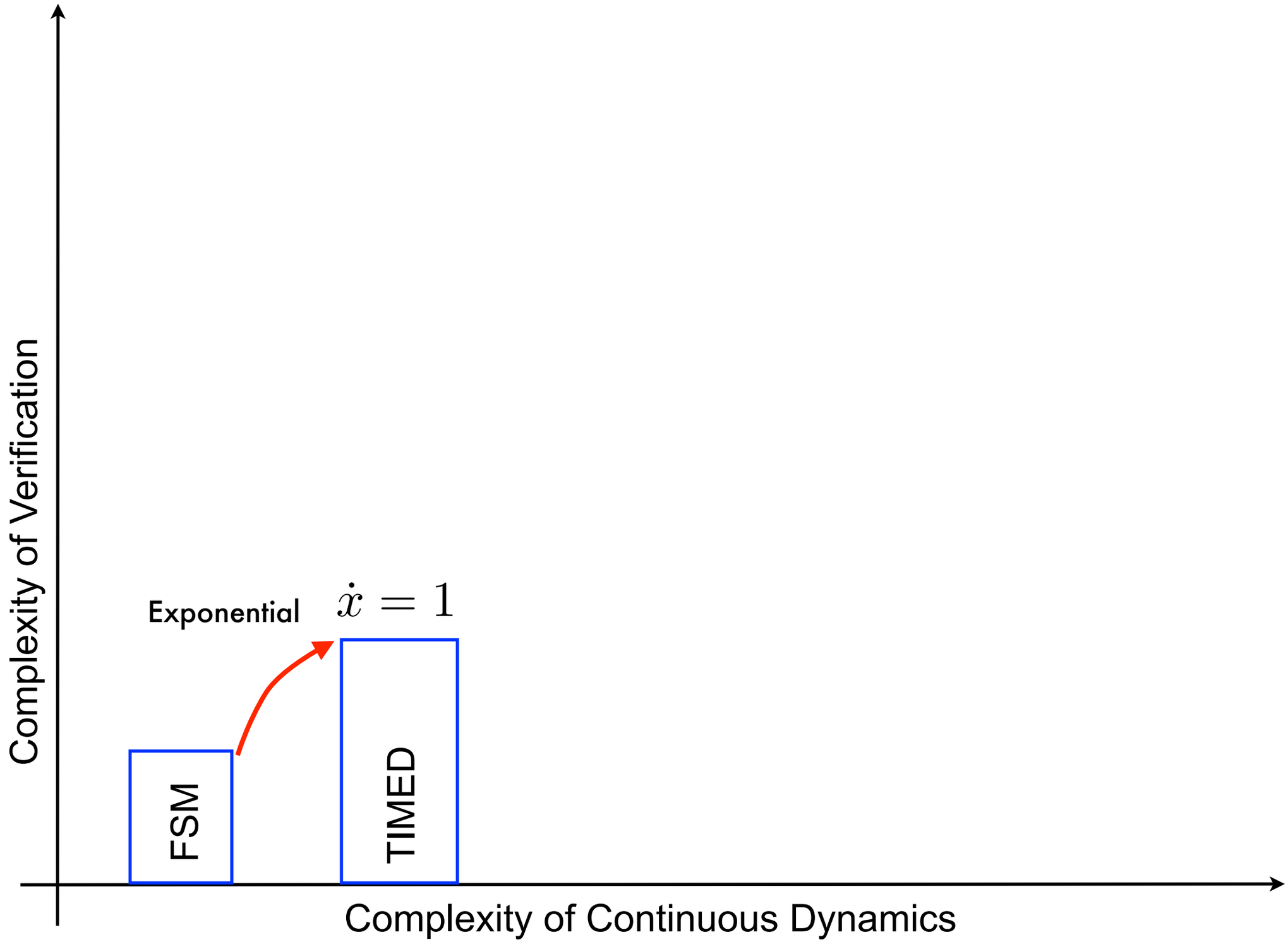

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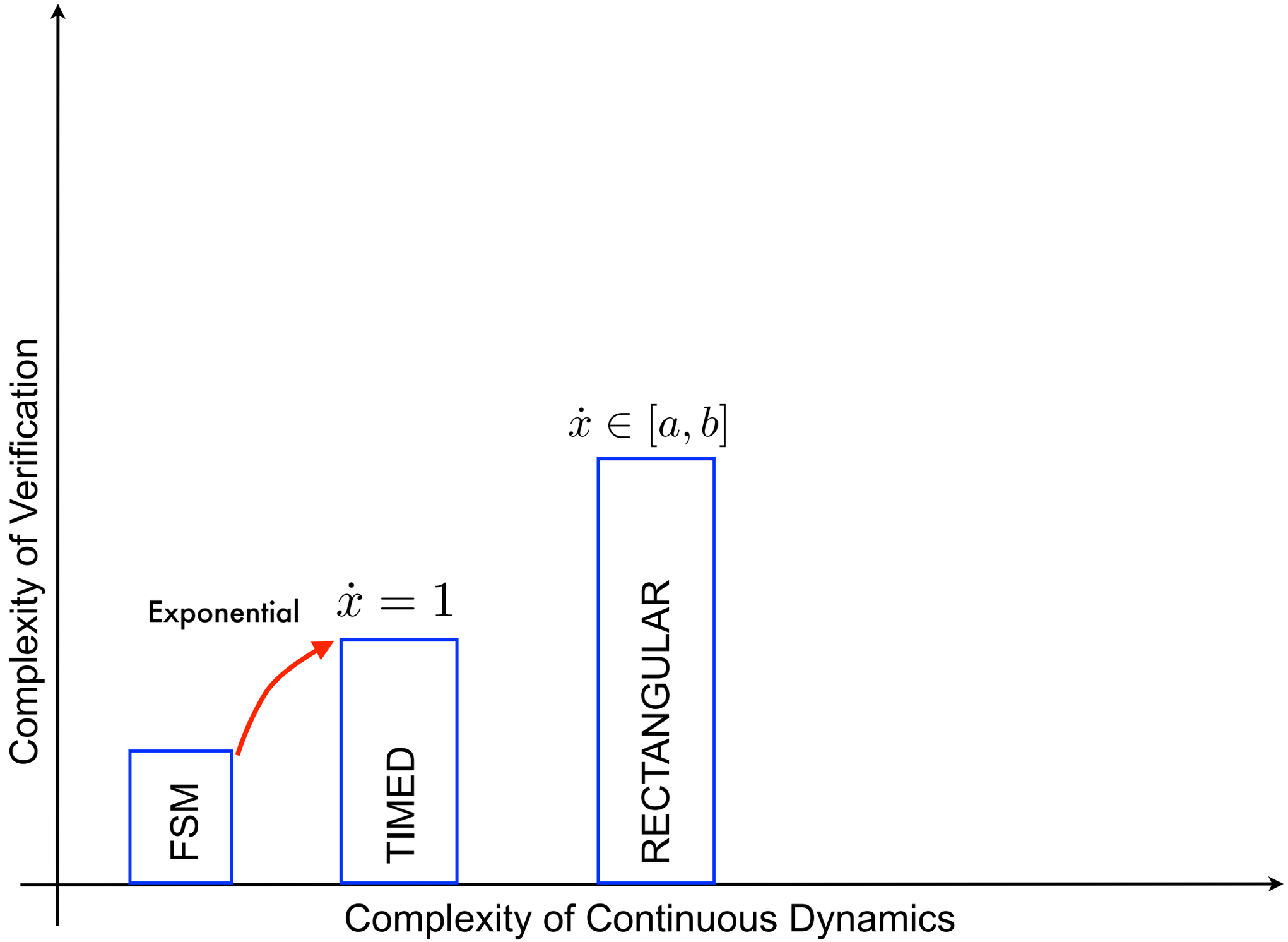

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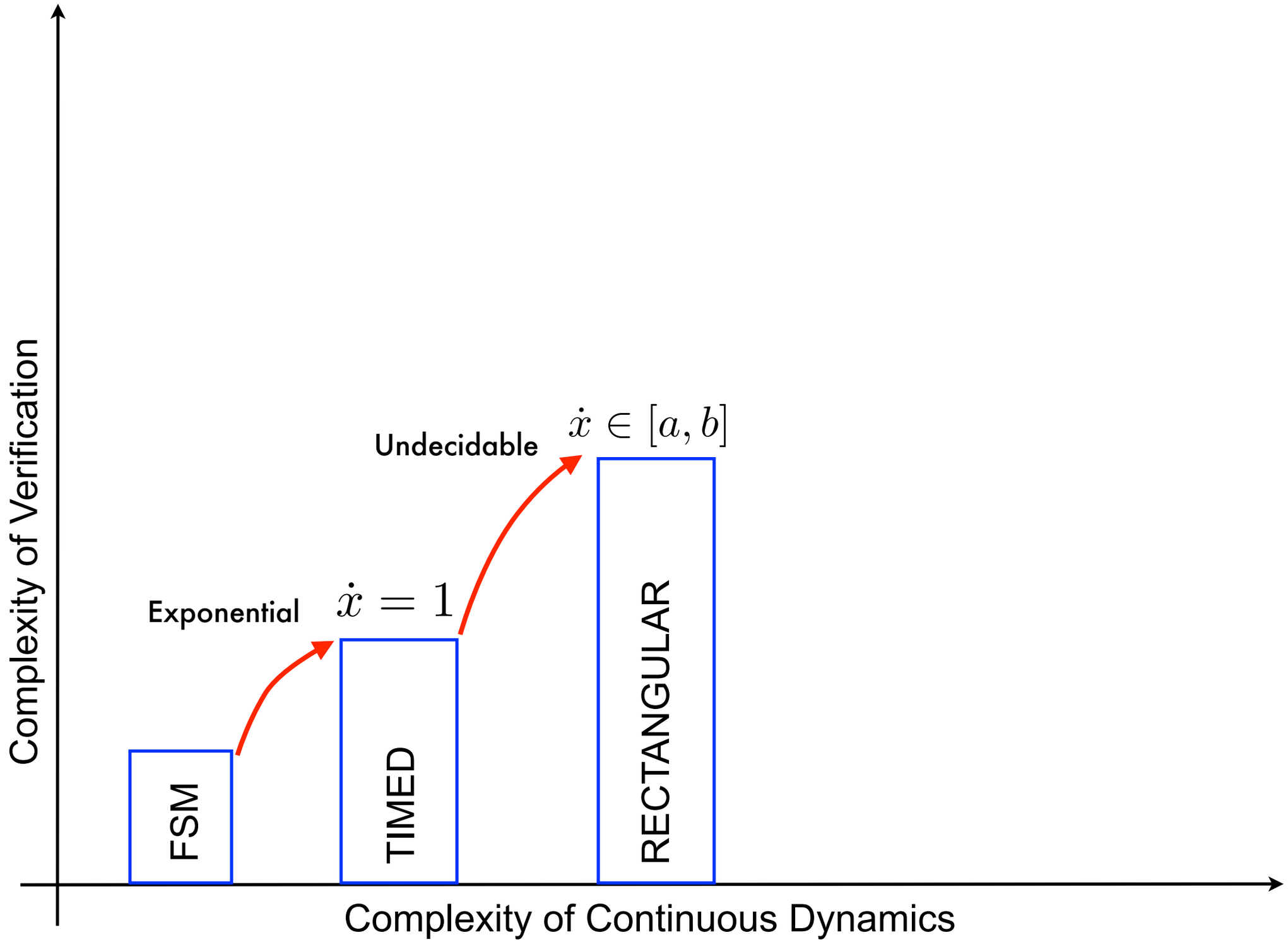

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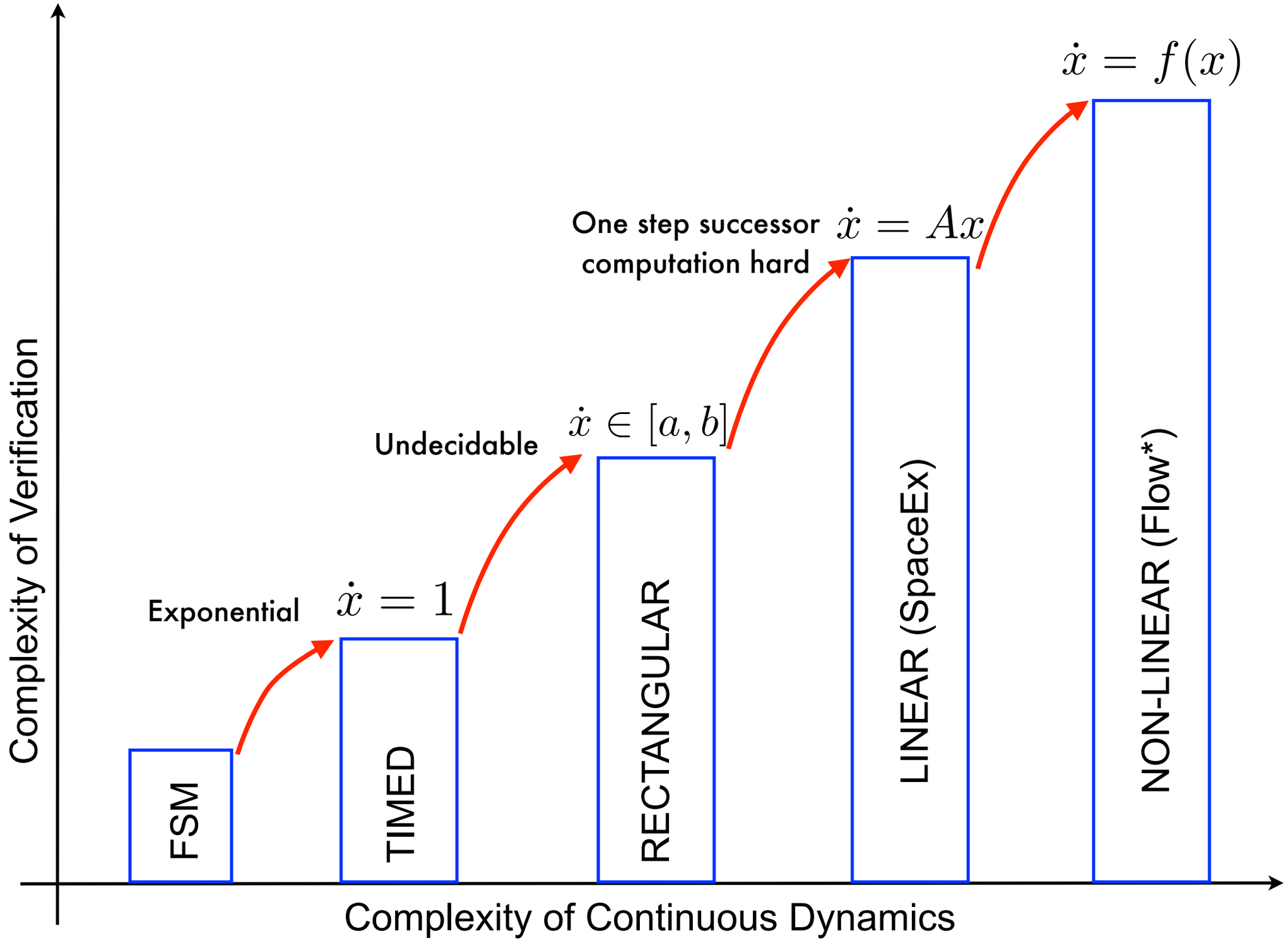


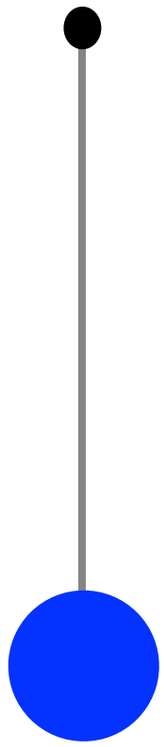
Overview of stability verification

Stability

Small changes to the initial state of the system result in small changes to the behavior of the system

- The controlled behavior of the car depends gracefully on small variations to its starting orientation

Stability in a pendulum



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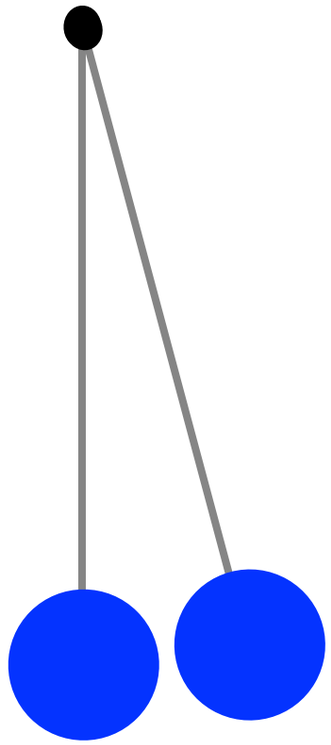
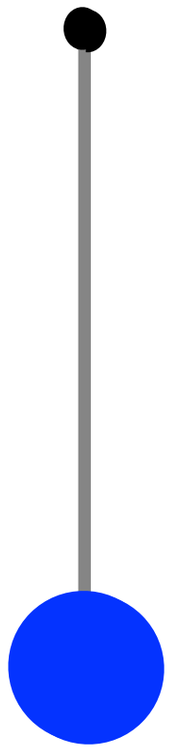
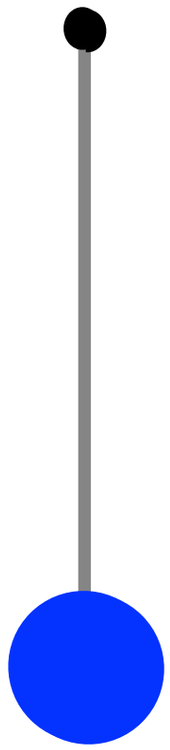


Stability in a pendulum



Stability in a pendulum



Stable Equilibrium

Stability in a pendulum

Stable Equilibrium

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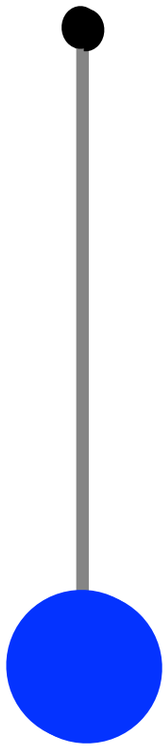

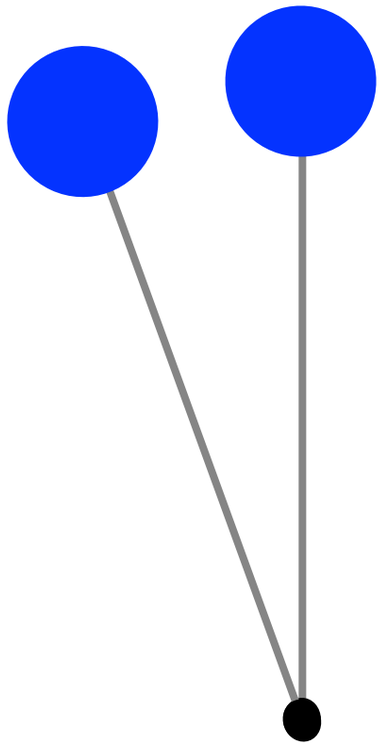

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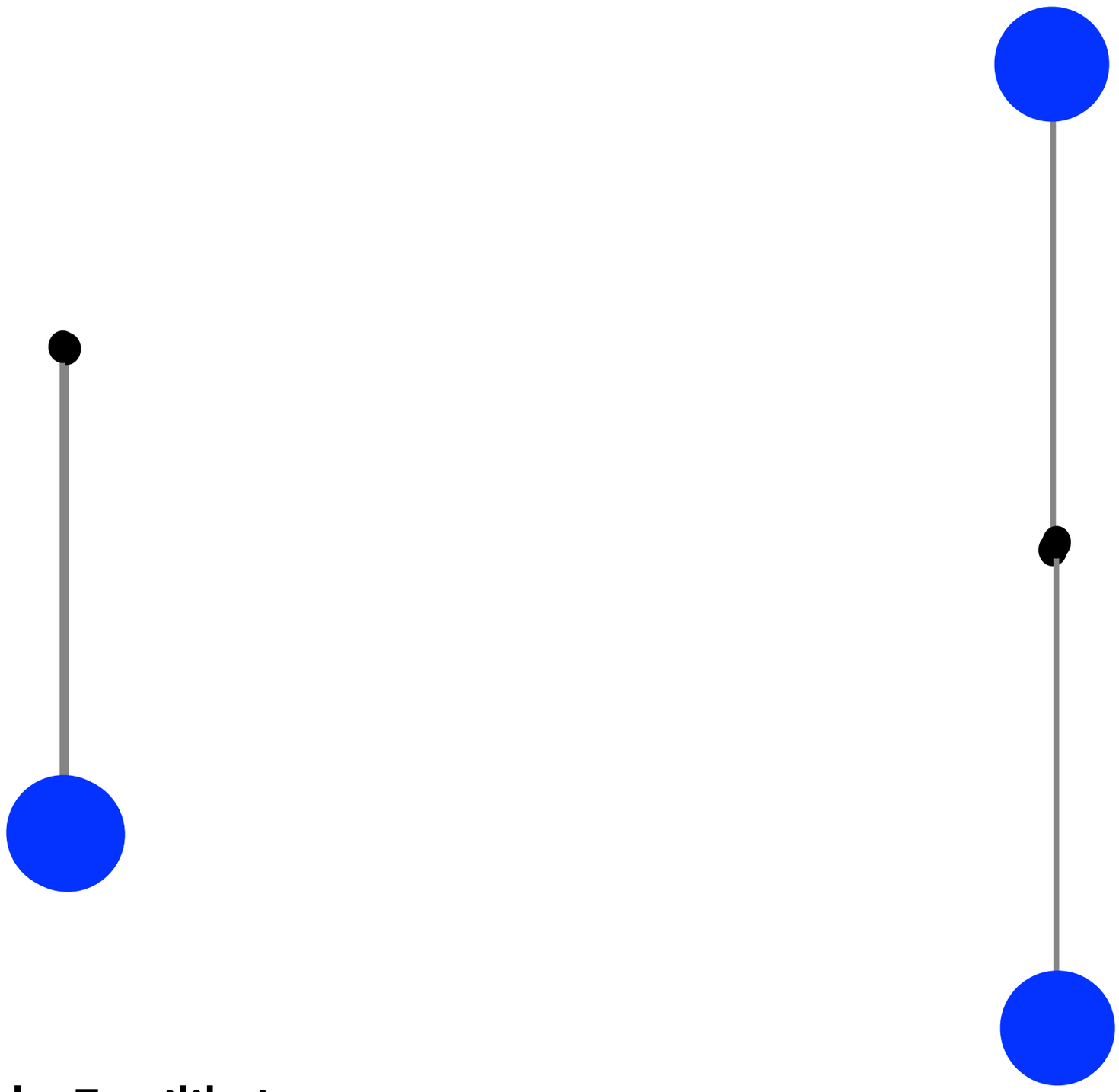

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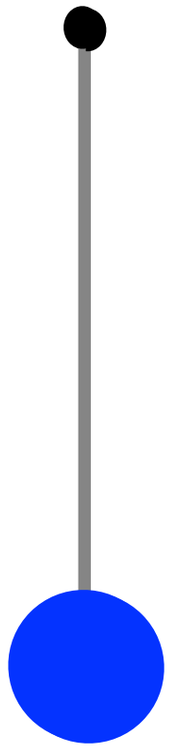

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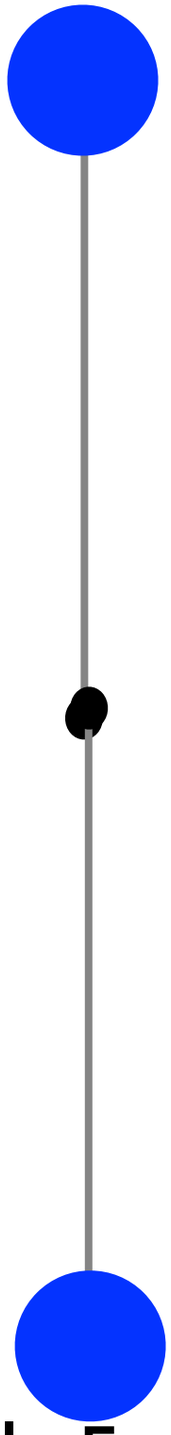
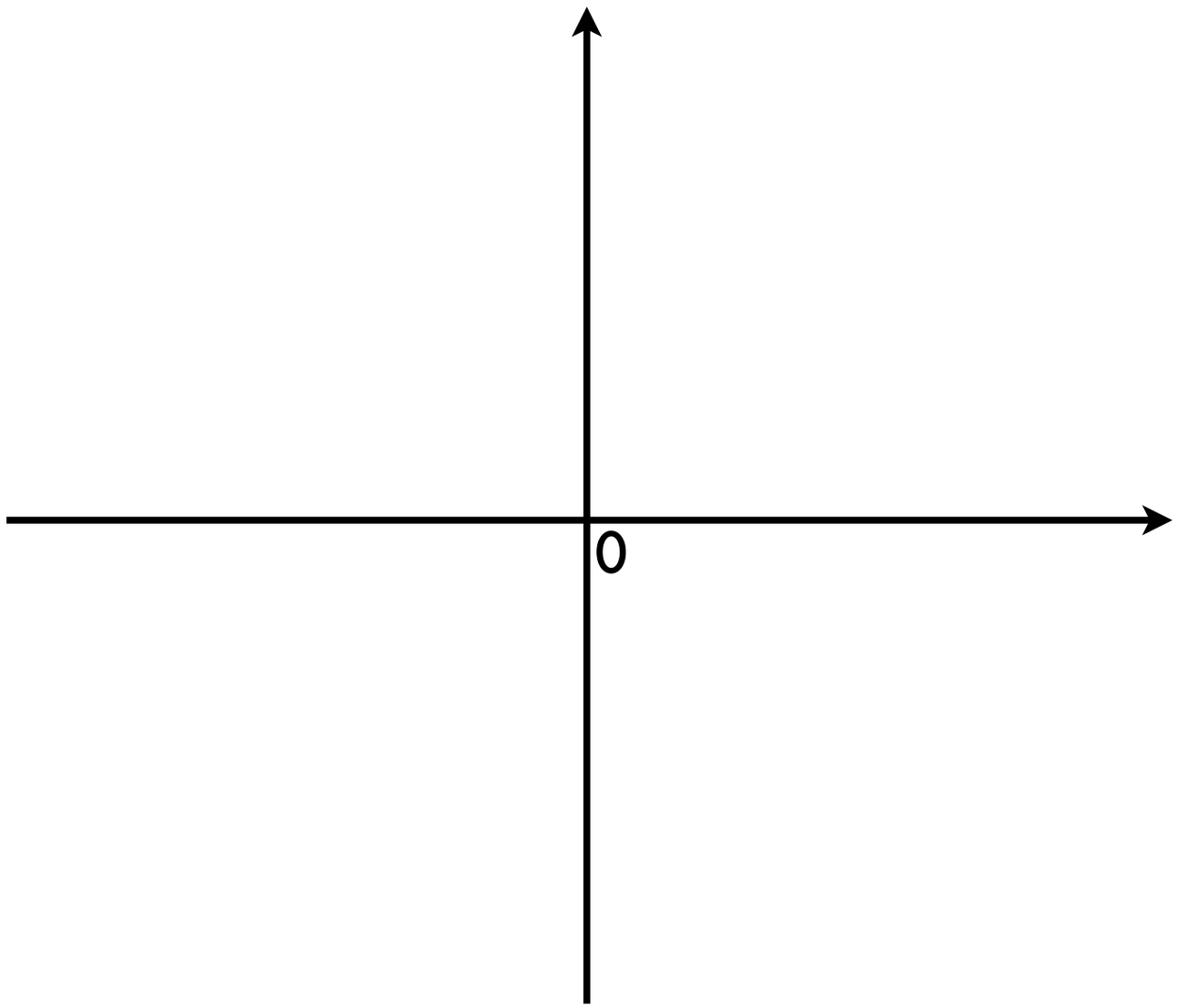


Unstable Equilibrium

Lyapunov stability



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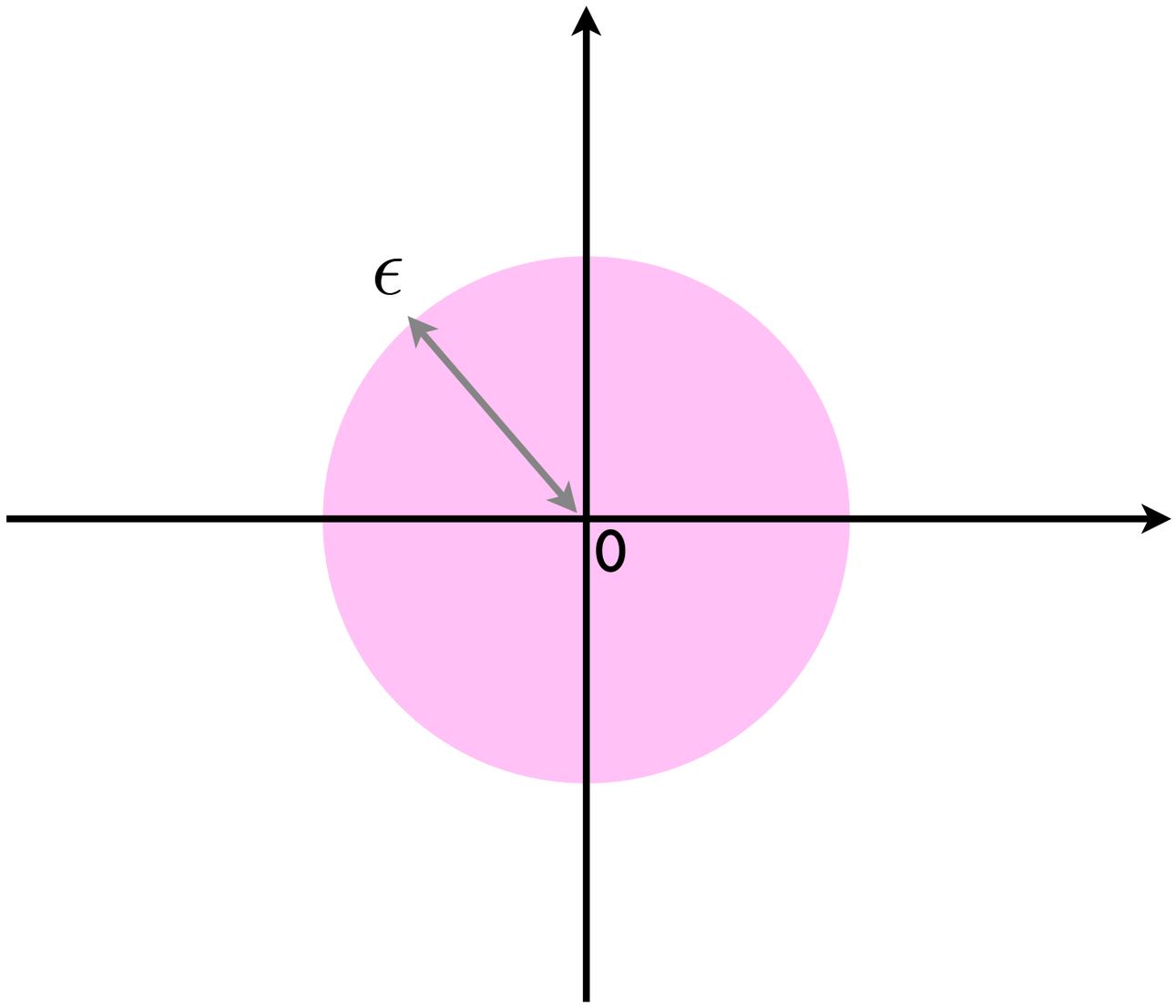

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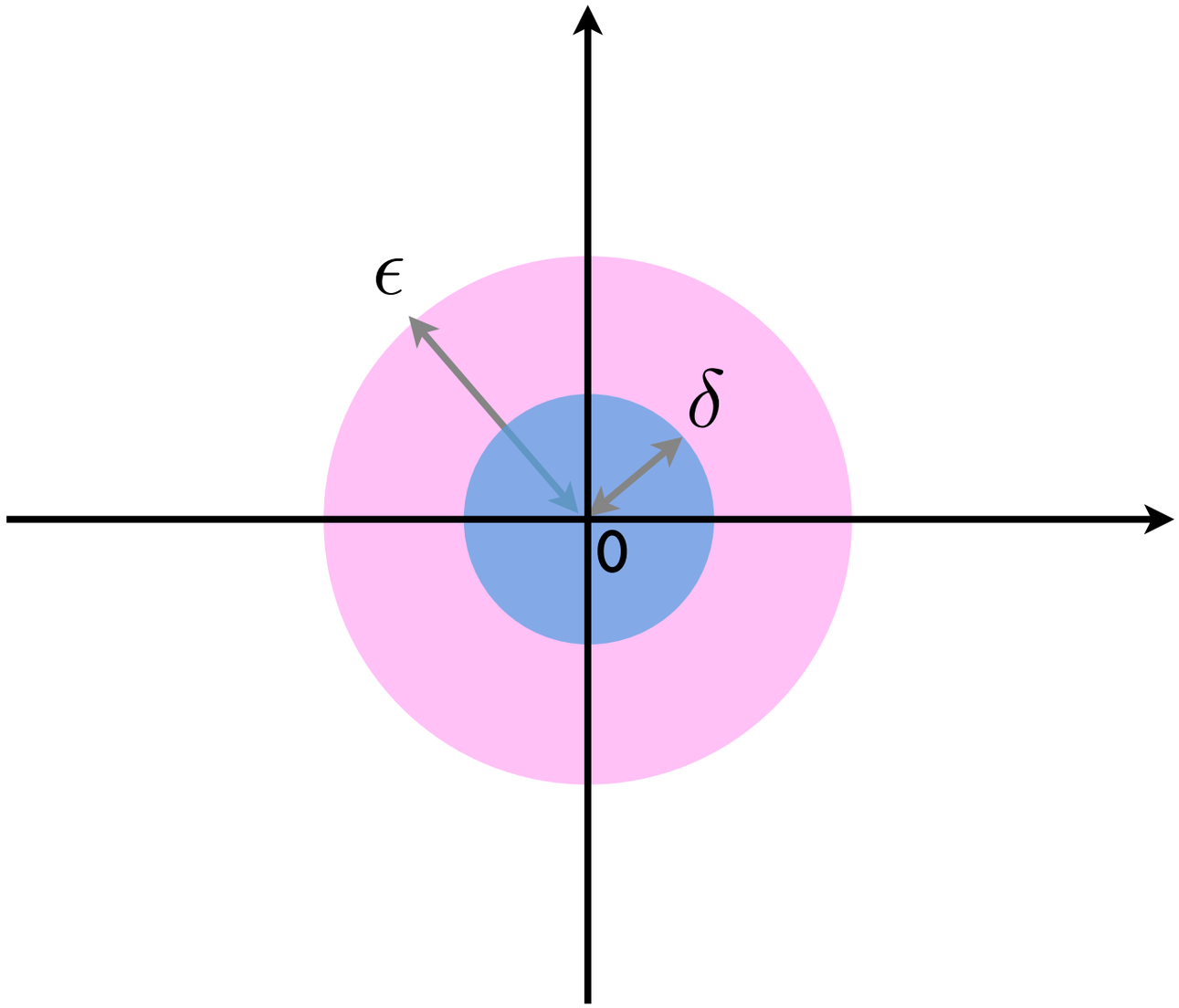

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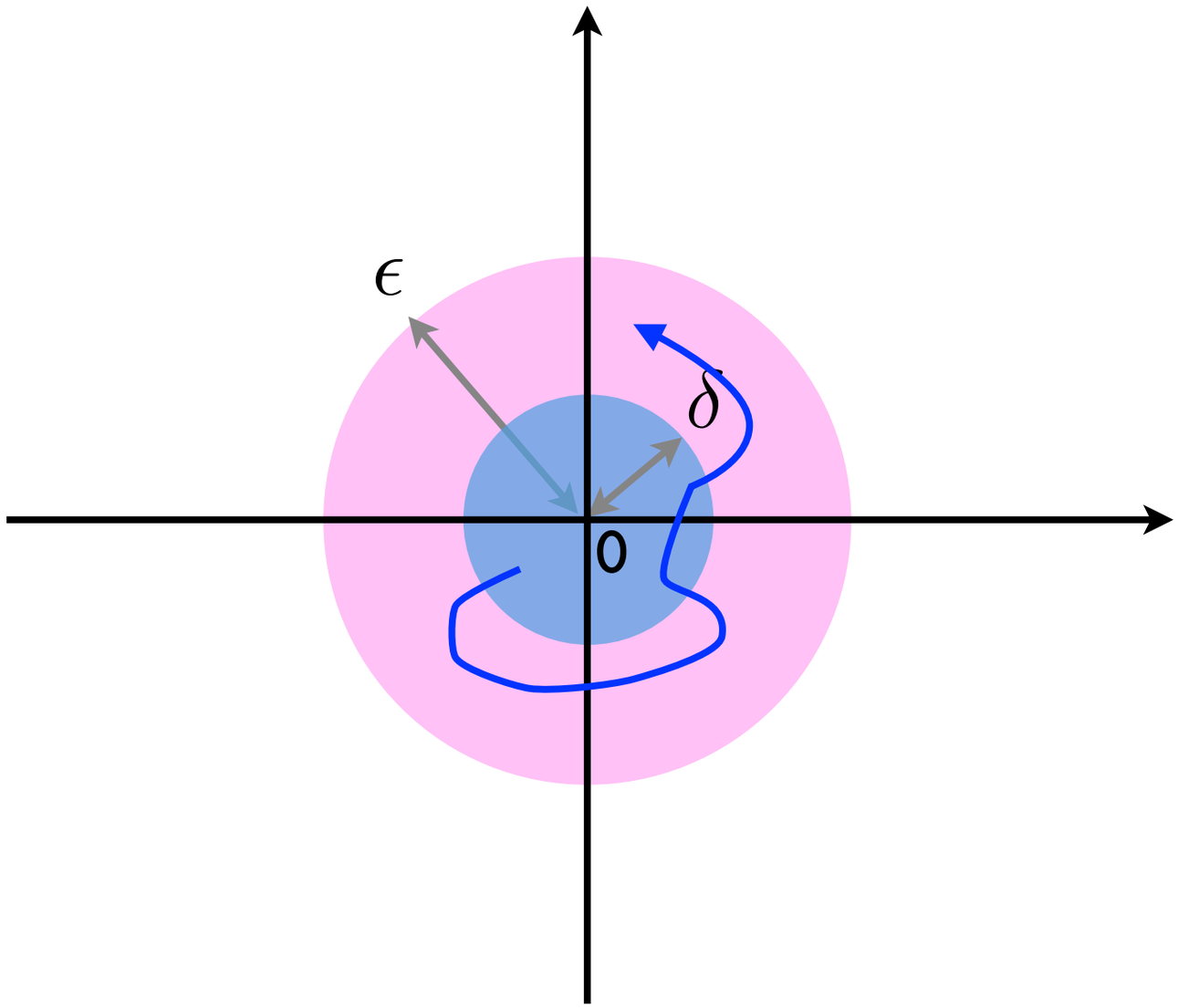

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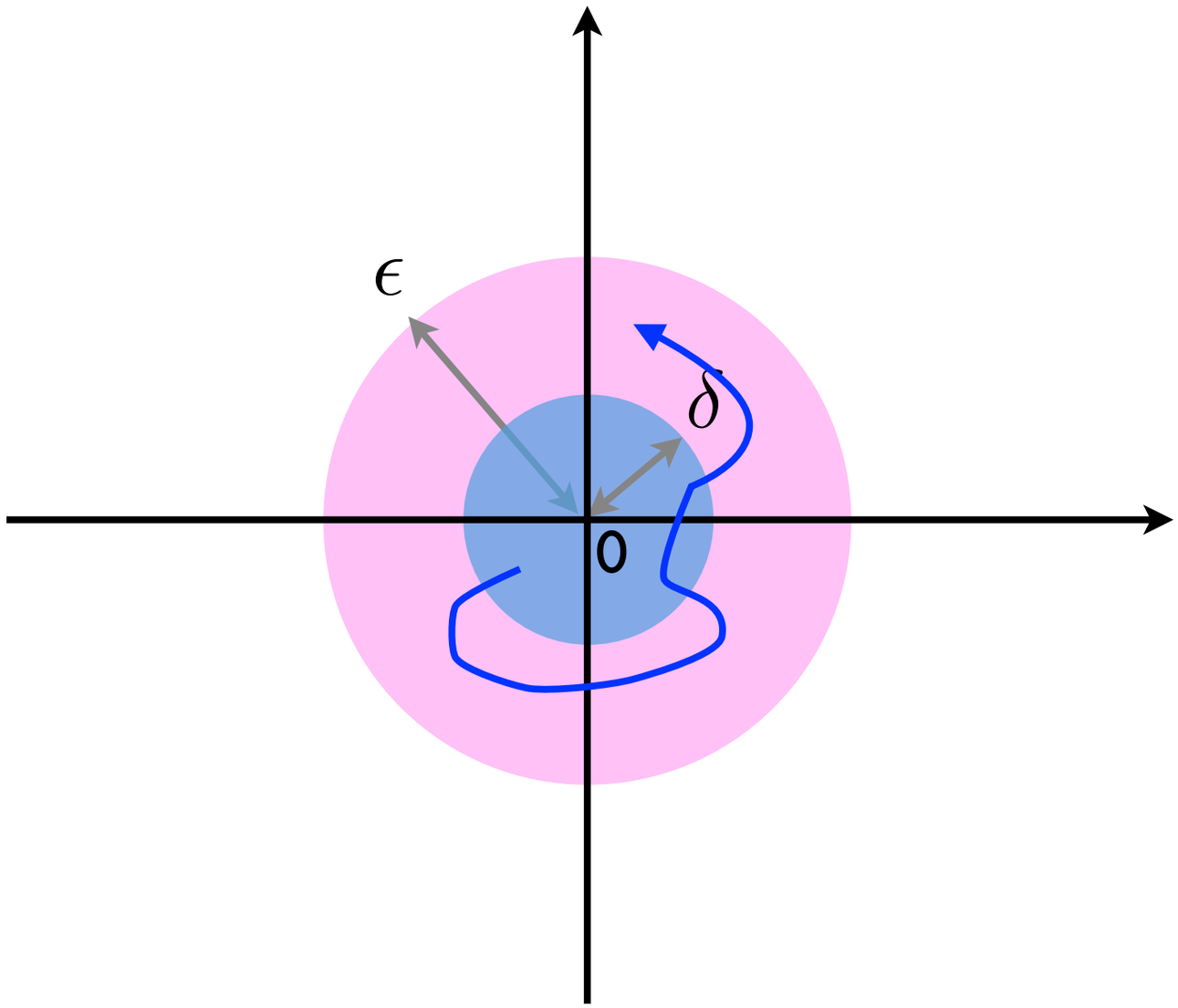
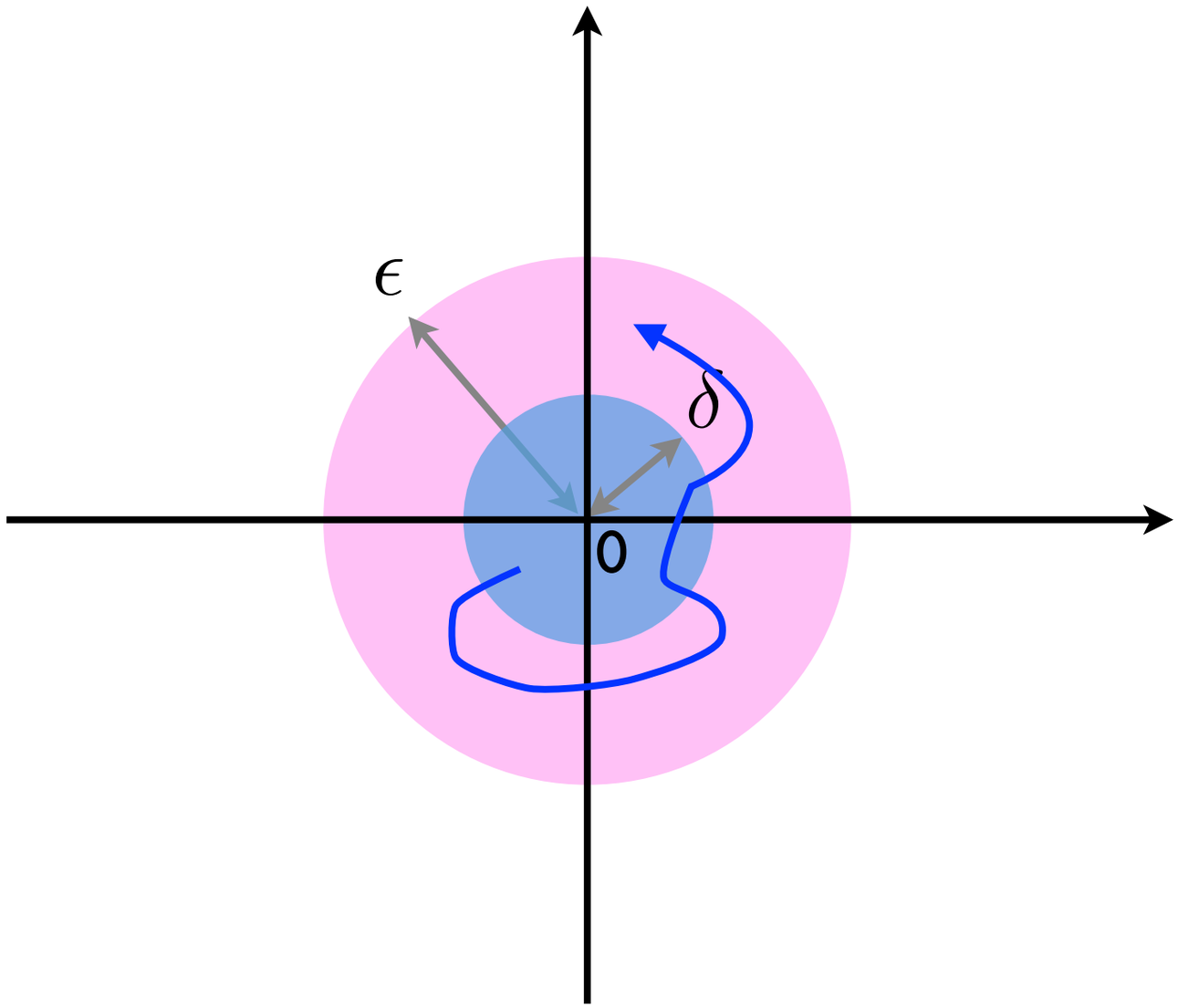


$$\forall \epsilon > 0 \, \exists \delta > 0 \, [(\sigma(0) \in B_\delta(0)) \Rightarrow \forall t (\sigma(t) \in B_\epsilon(0))]$$

Lyapunov stability



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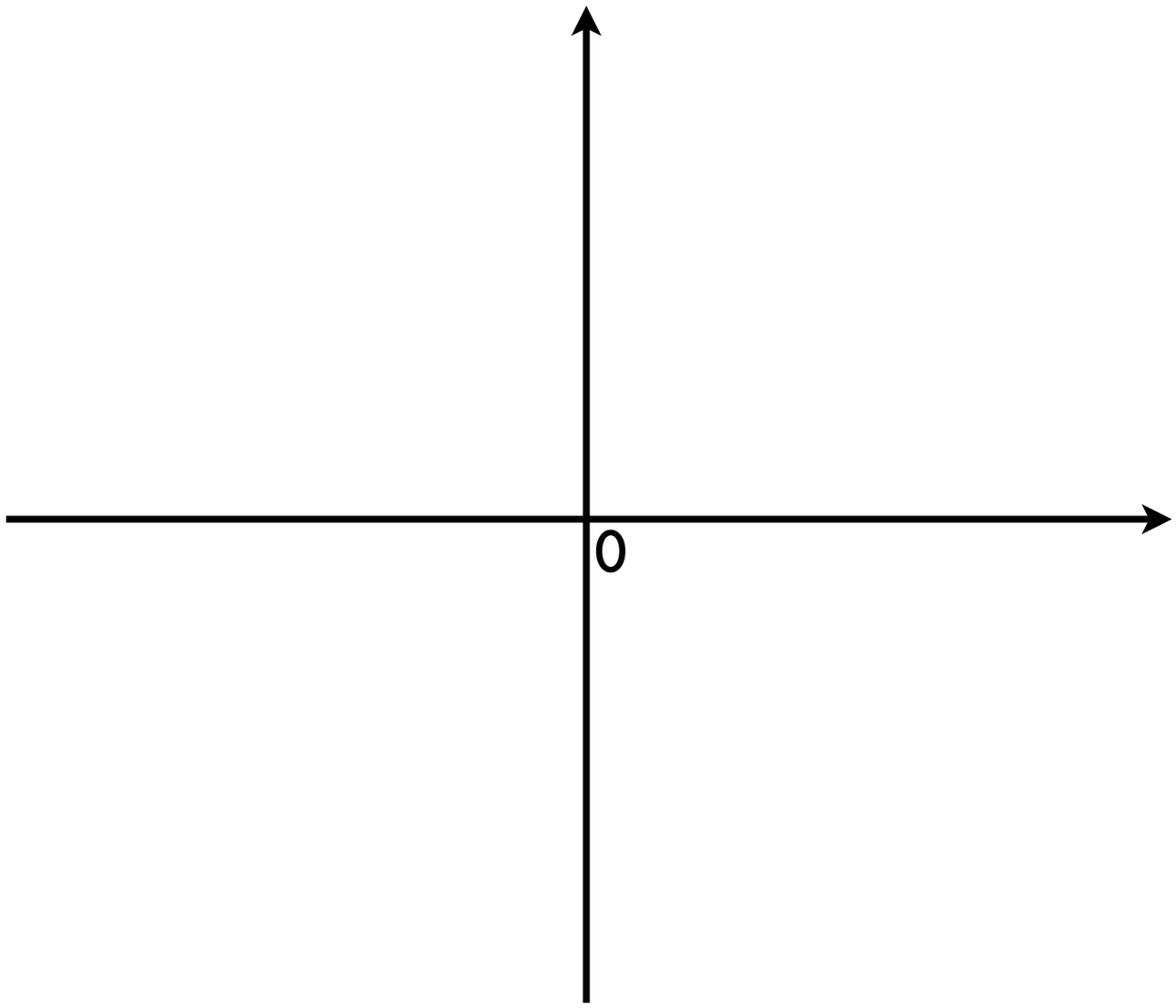
"Continuity of the transition relation at the origin"

Asymptotic stability

"Lyapunov stability + Convergence"



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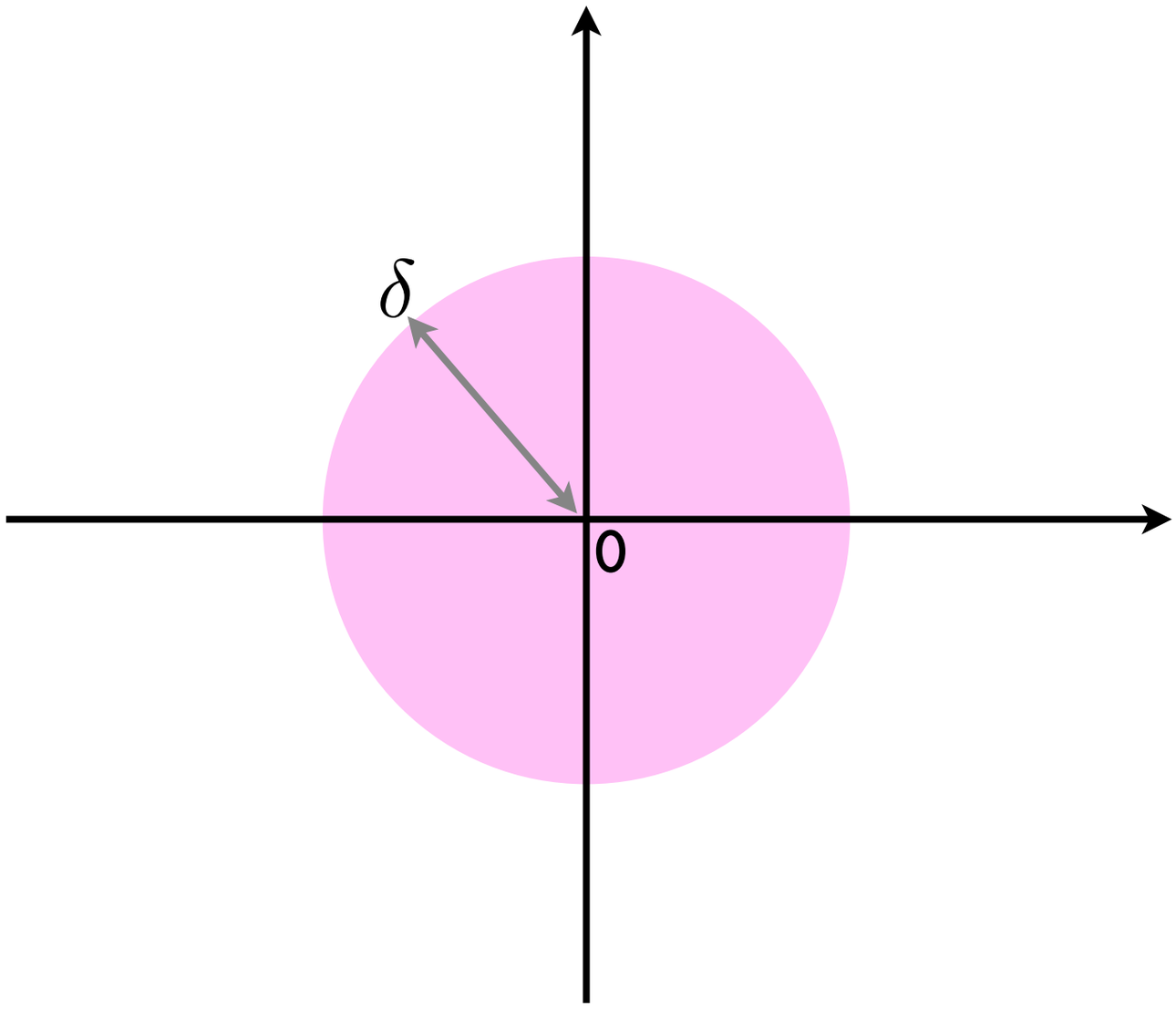

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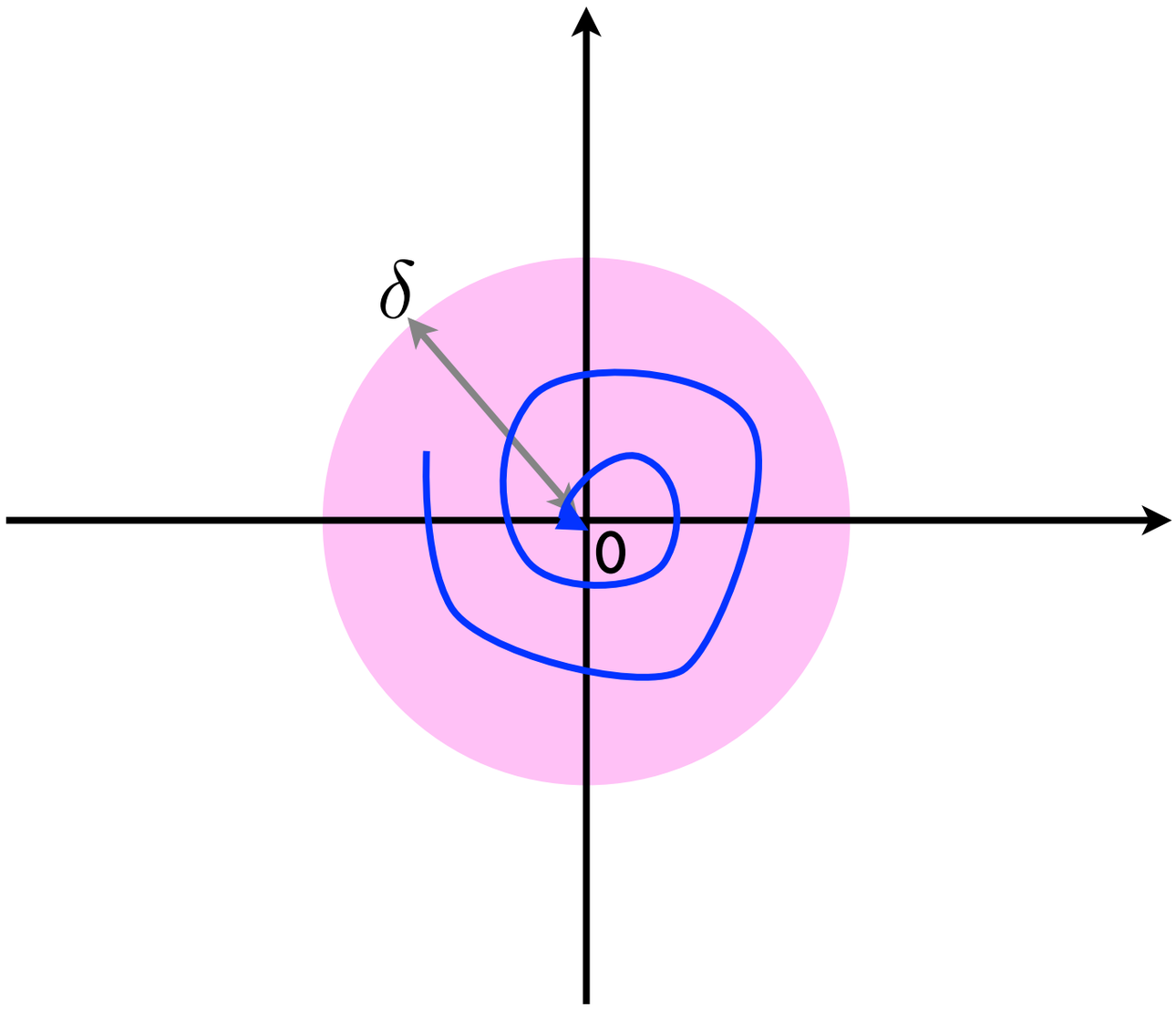

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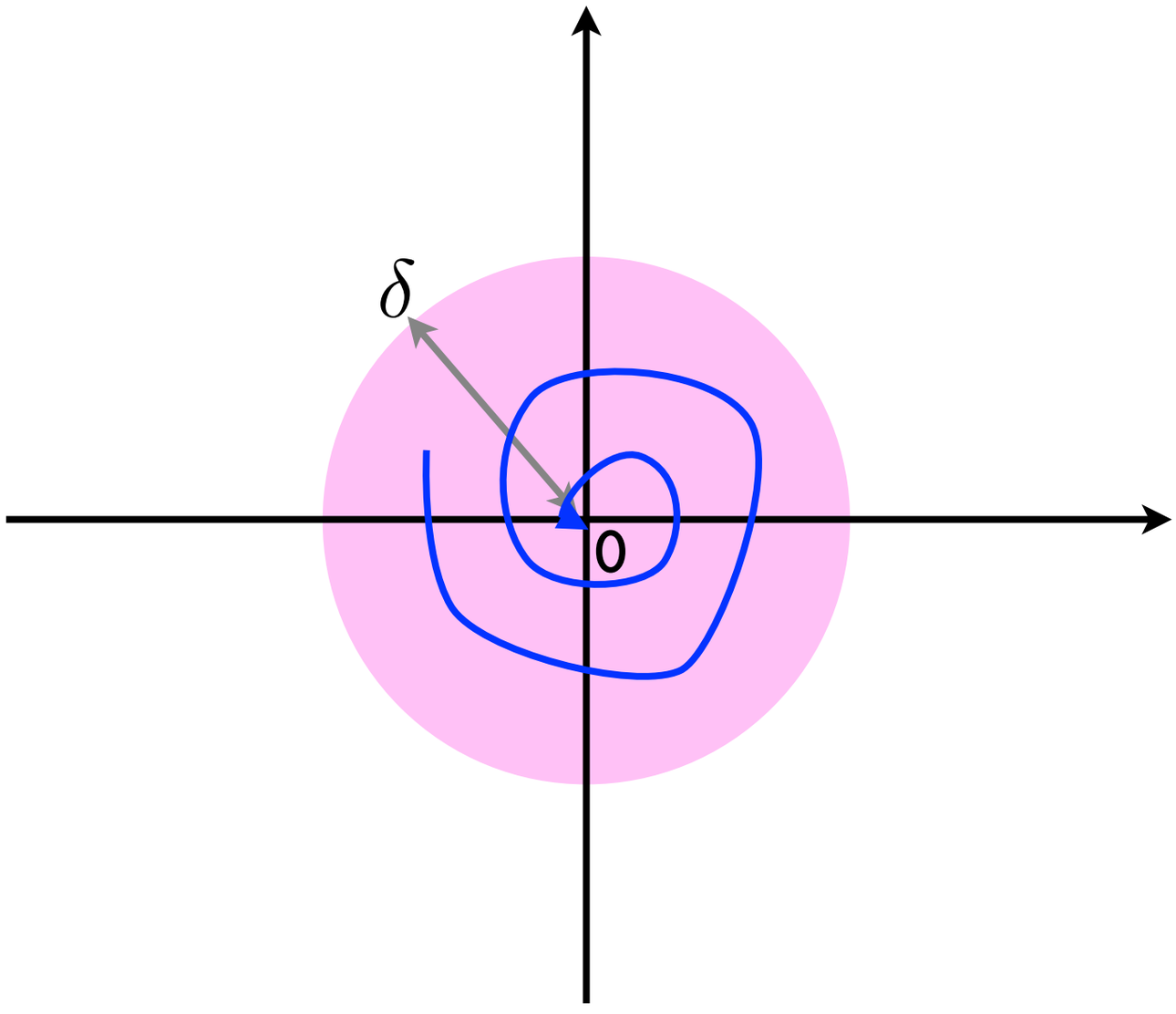

$$\exists \delta > 0 \left[(\sigma(0) \in B_\delta(0)) \Rightarrow Converge(\sigma, 0) \right]$$

$$Converge(\sigma, 0) \equiv \forall \epsilon > 0, \exists T \geq 0, \sigma(t) \in B_\epsilon(0), \forall t \geq T$$

Linear dynamical systems

$$\dot{x}(t) = ax(t)$$

$$x(t) = e^{at}x(0)$$

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If a is 0, $x(t)$ is always $x(0)$

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Eigen value: λ such that there exists $x \neq 0$ with $Ax = \lambda x$

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all the eigen values of A have negative real parts

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The linear system is unstable if

A has at least one eigen value with positive real parts

Non-linear systems

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Lyapunov's first method - Linearization

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System $\dot{x} = F(x)$

Equilibrium $F(0) = 0$

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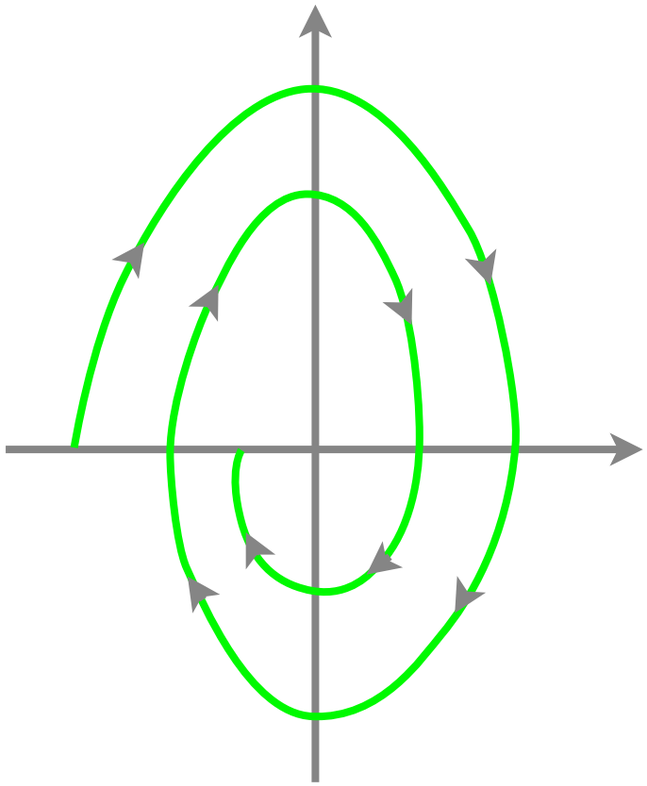
If A has is hyperbolic (all eigen values have non-zero real part), then stability of the original system is equivalent to the stability of the linearization.

Linear Switched Systems

Linear Switched Systems



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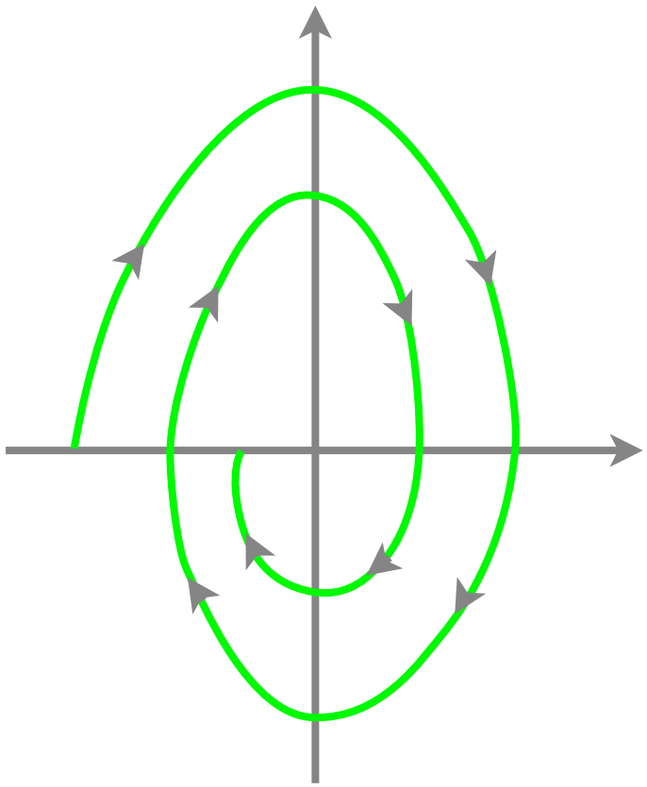

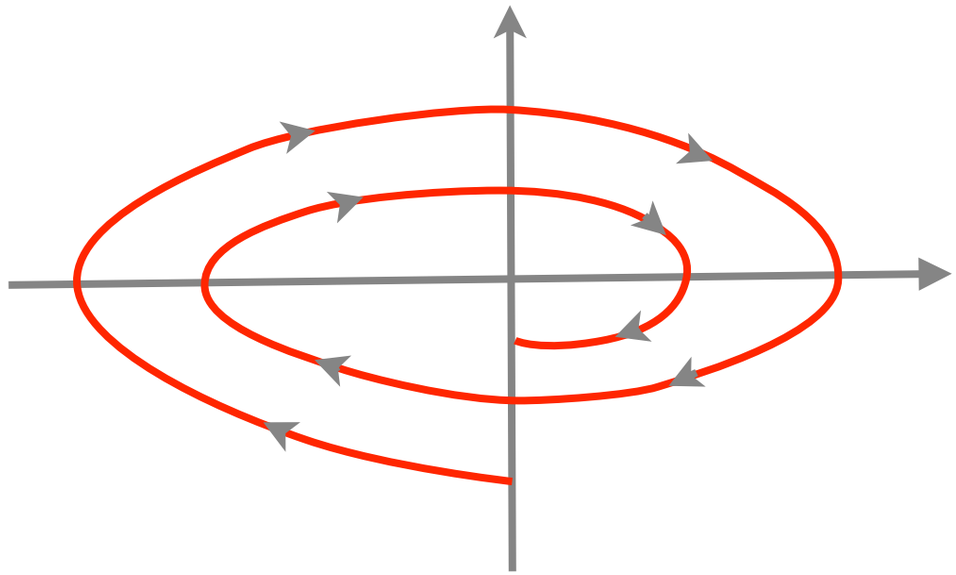

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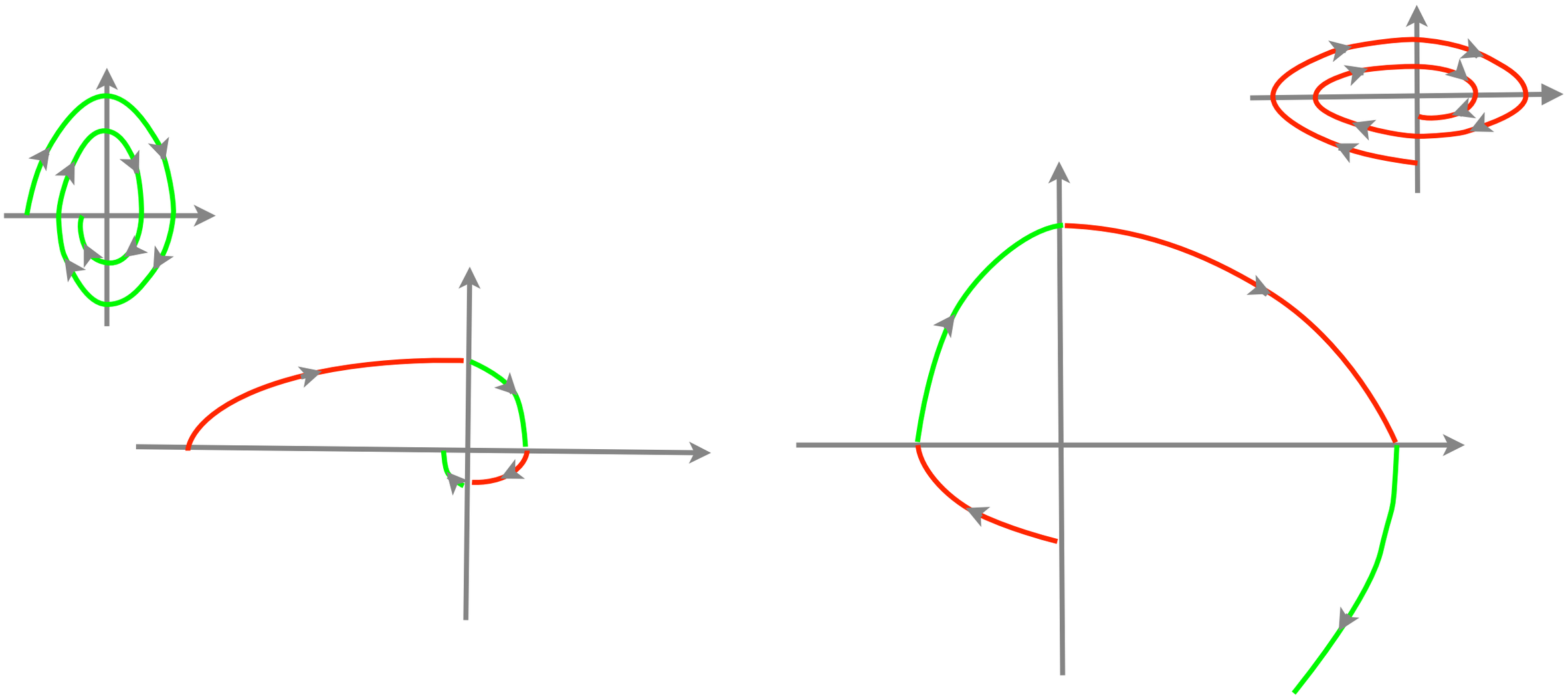

Eigen value analysis doesn't extend to mixed discrete continuous systems

Lyapunov's Second Method

System	$\dot{x} = F(x)$
Equilibrium	$F(0) = 0$
Solution	$\varphi(x, t)$

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If there exists a Lyapunov function for the system,

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Lyapunov function: Illustration

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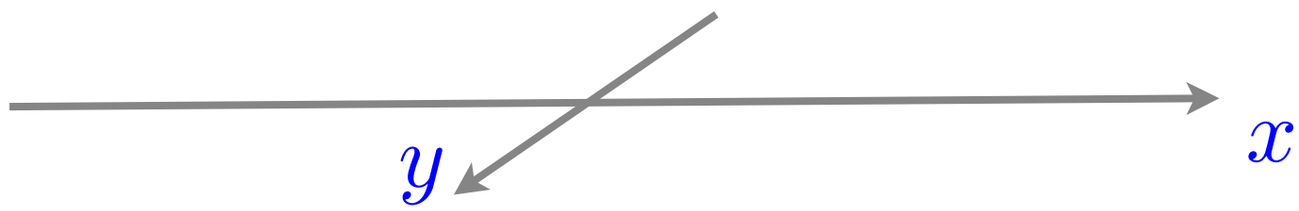

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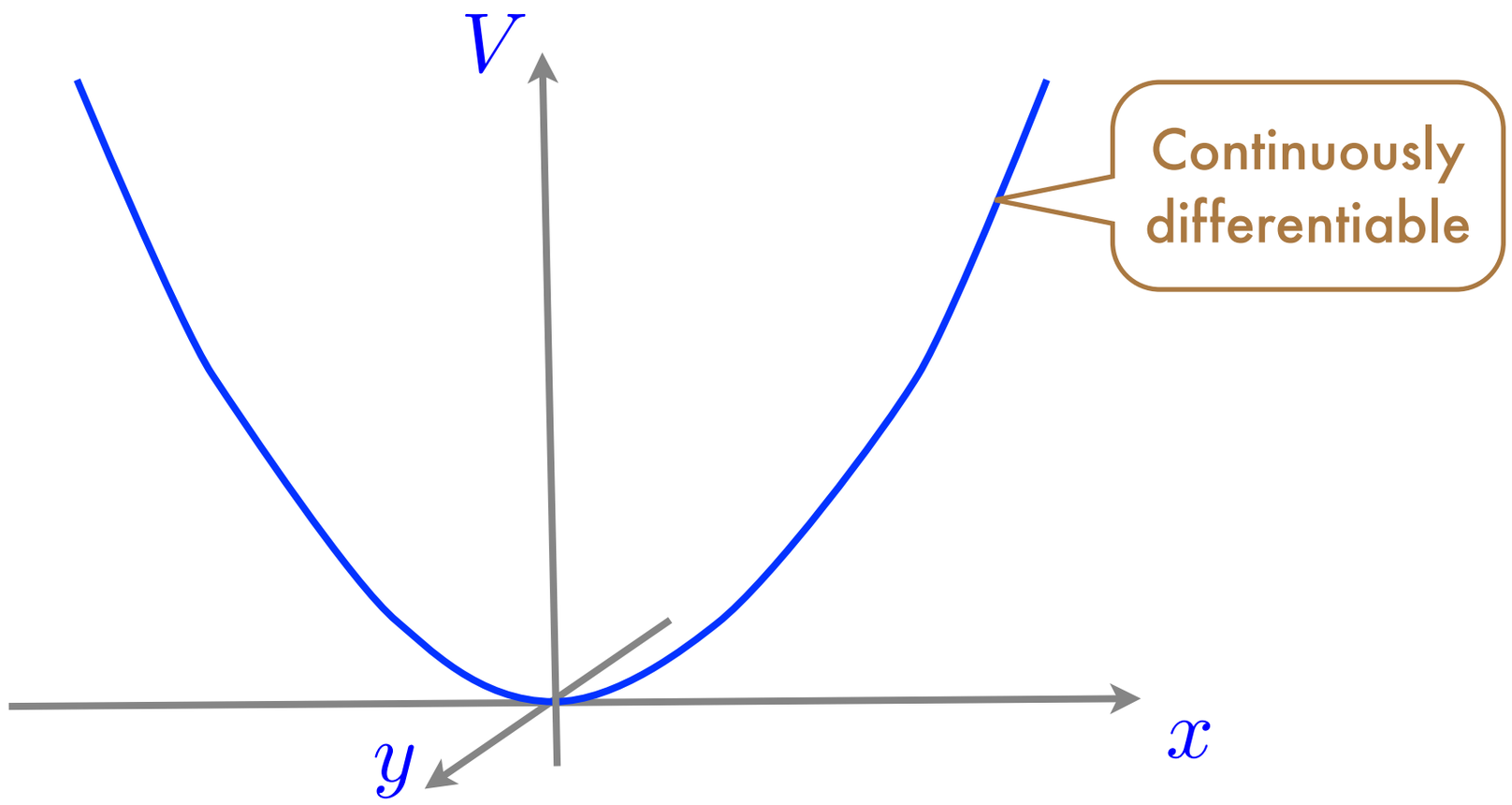

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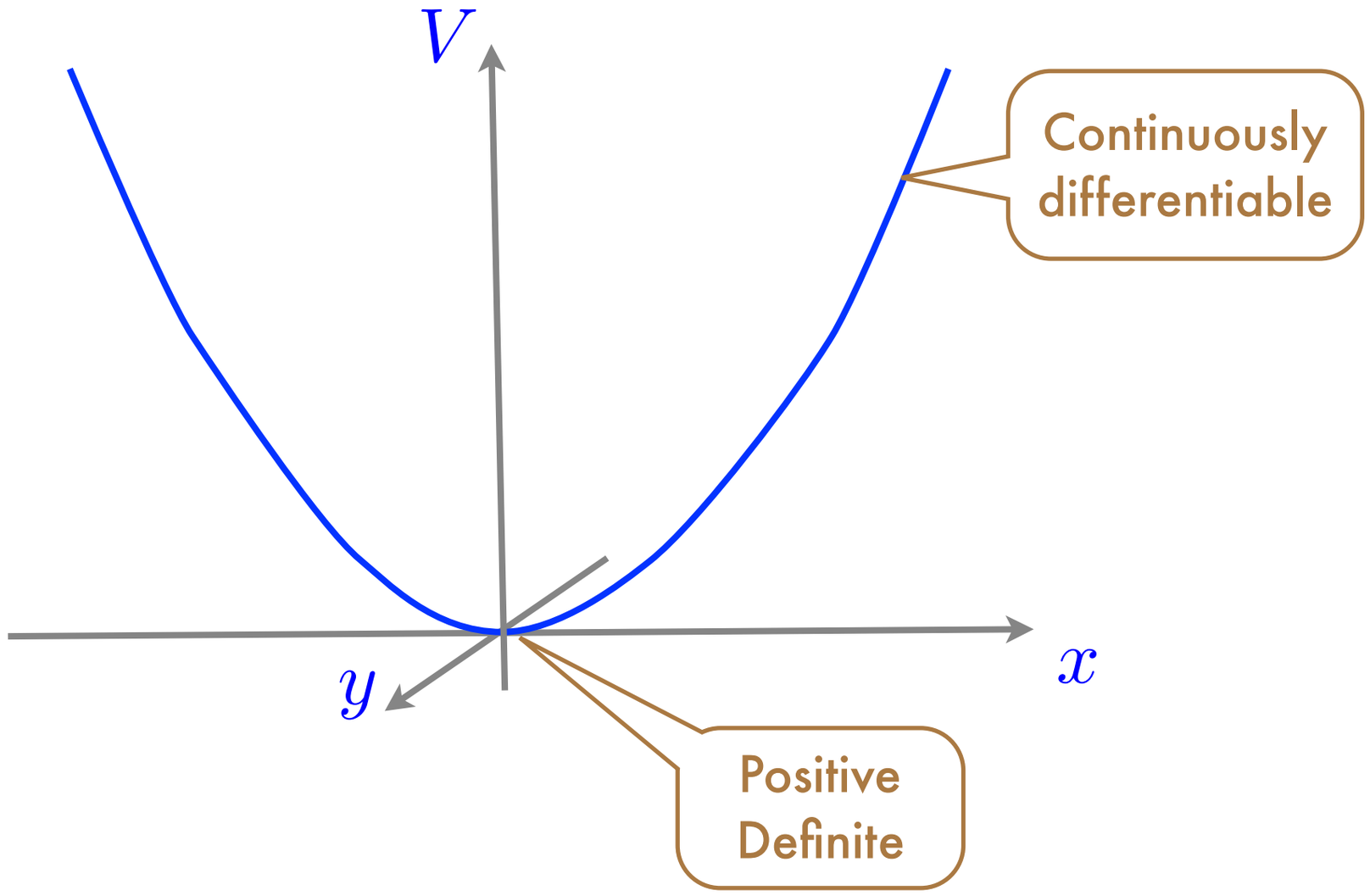

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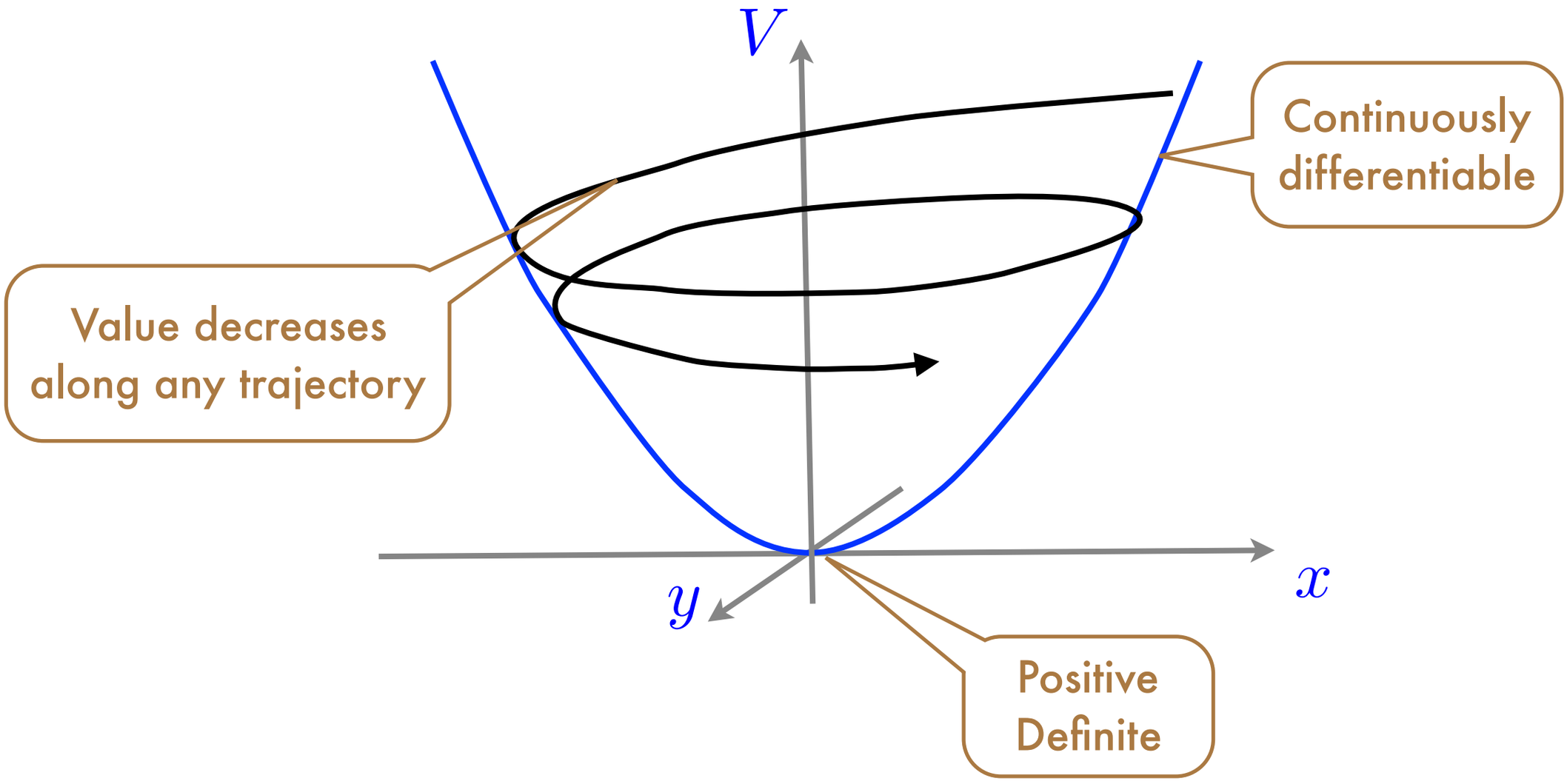

Lyapunov function

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Continuously differentiable

$V : \mathbb{R}^n \to \mathbb{R}+$

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System $\dot{x} = -x$

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- A quadratic Lyapunov function exists for every stable linear system
- It can be computed by solving a linear matrix inequality
- Such complete results do not exist for non-linear systems or hybrid systems (even in the linear case)

Lyapunov functions for hybrid systems

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- Common Lyapunov function - a function which is a Lyapunov function for each mode of the system

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- *Reference: Switching in Systems and Control - Daniel Liberzon*

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Lyapunov function: Computation

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Polynomial Template

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Check if there exist coefficients for which the polynomial is a sum-of-squares (SOS)

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Express again as a sum-of-squares constraint

Abstraction refinement for stability

Abstraction refinement for stability

- An abstraction refinement framework for a systematic proof search

Abstraction refinement for stability

- An abstraction refinement framework for a systematic proof search
- Notions of abstraction not well-studied

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Abstraction refinement for stability

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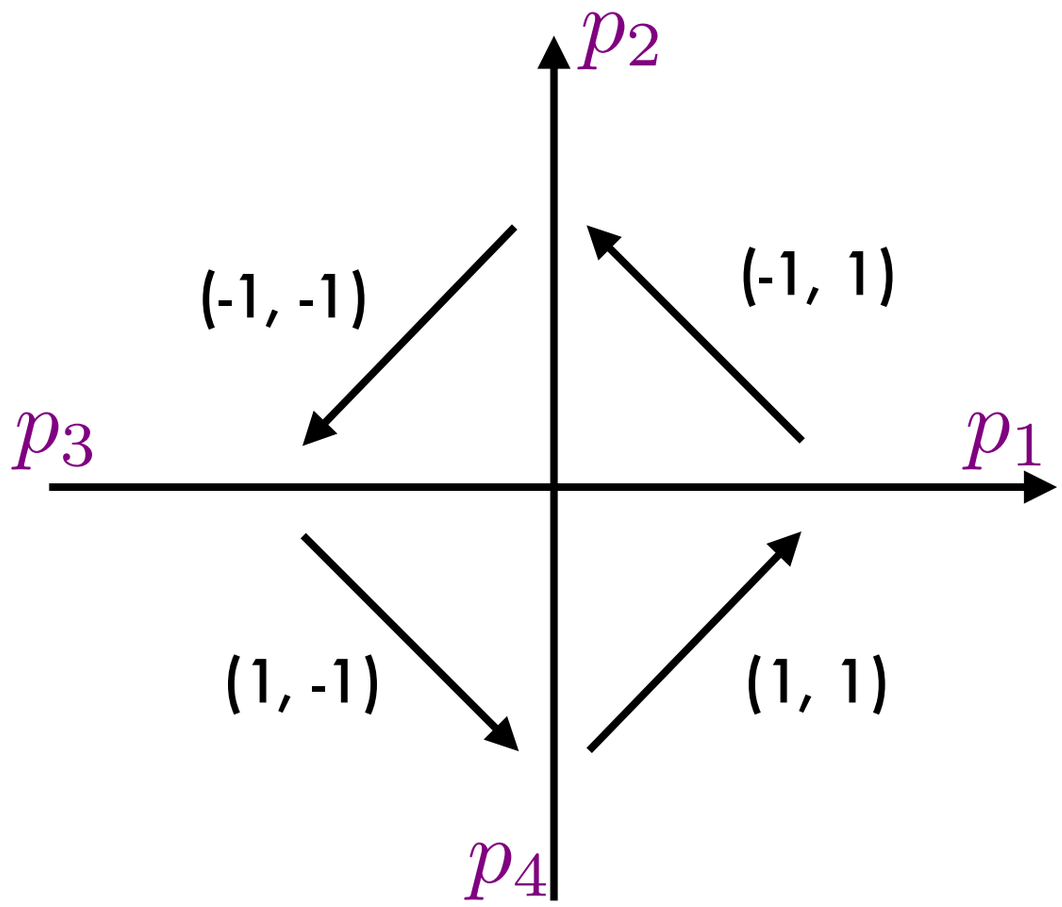
- Notions of abstraction not well-studied
- Do the discrete abstraction techniques for safety work? No!
- Modified predicate abstraction

Piecewise Constant Derivative System



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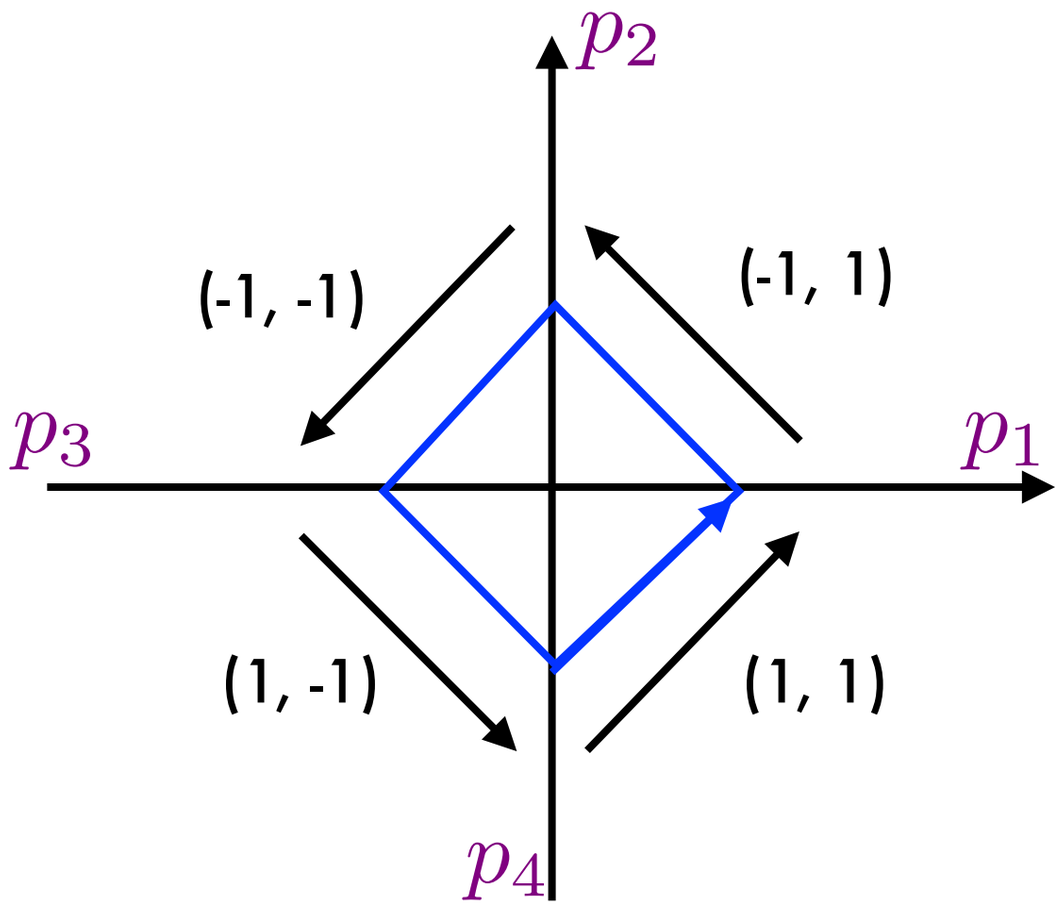

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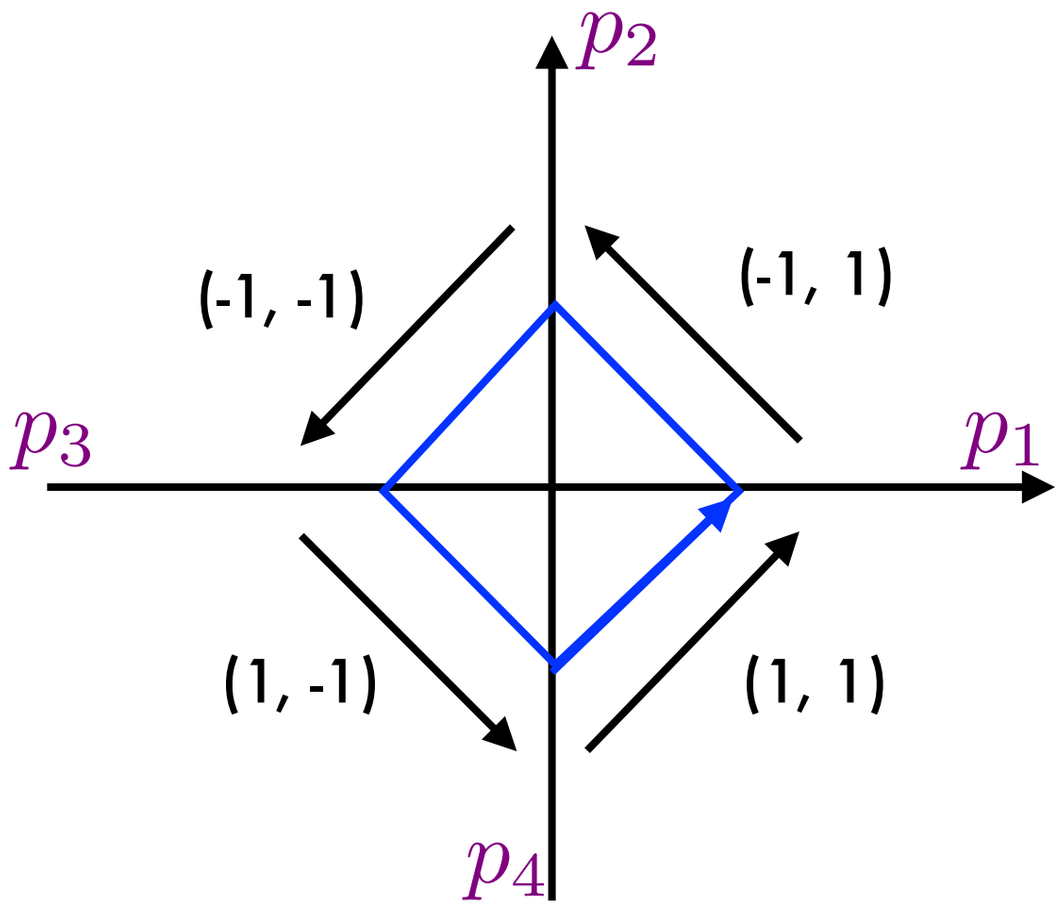
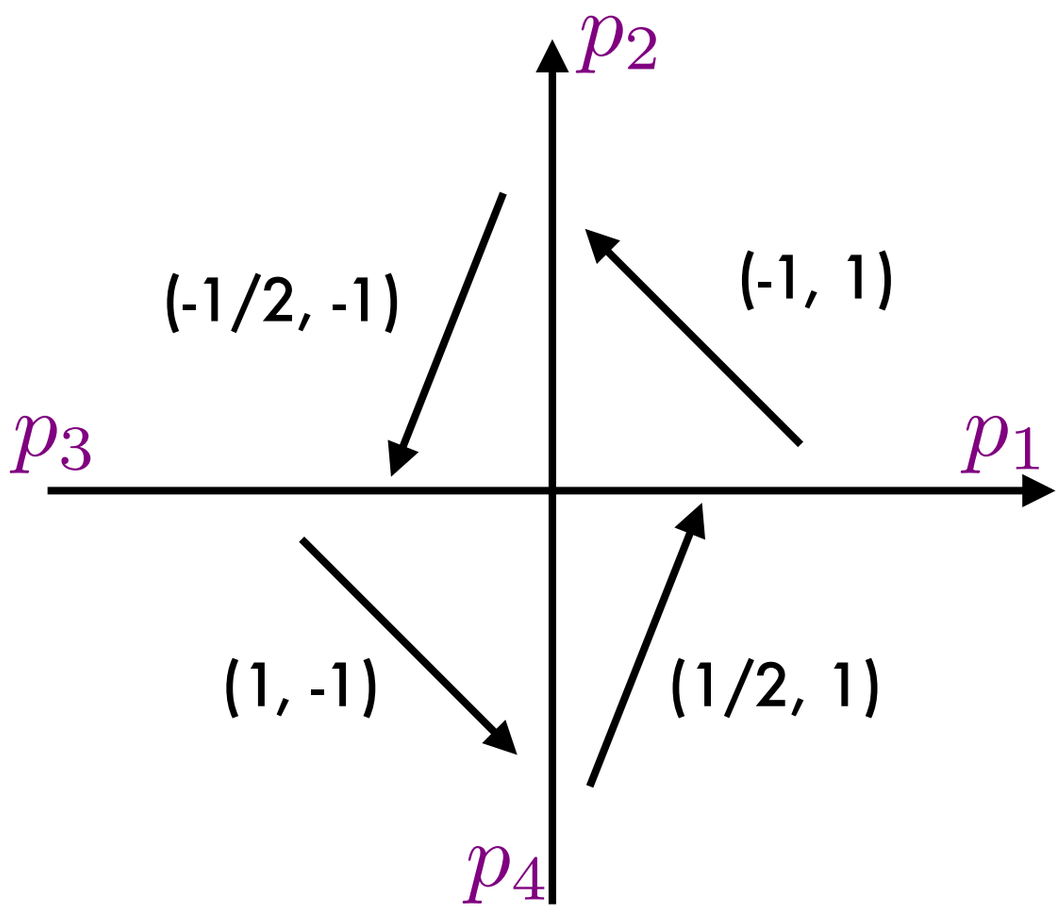
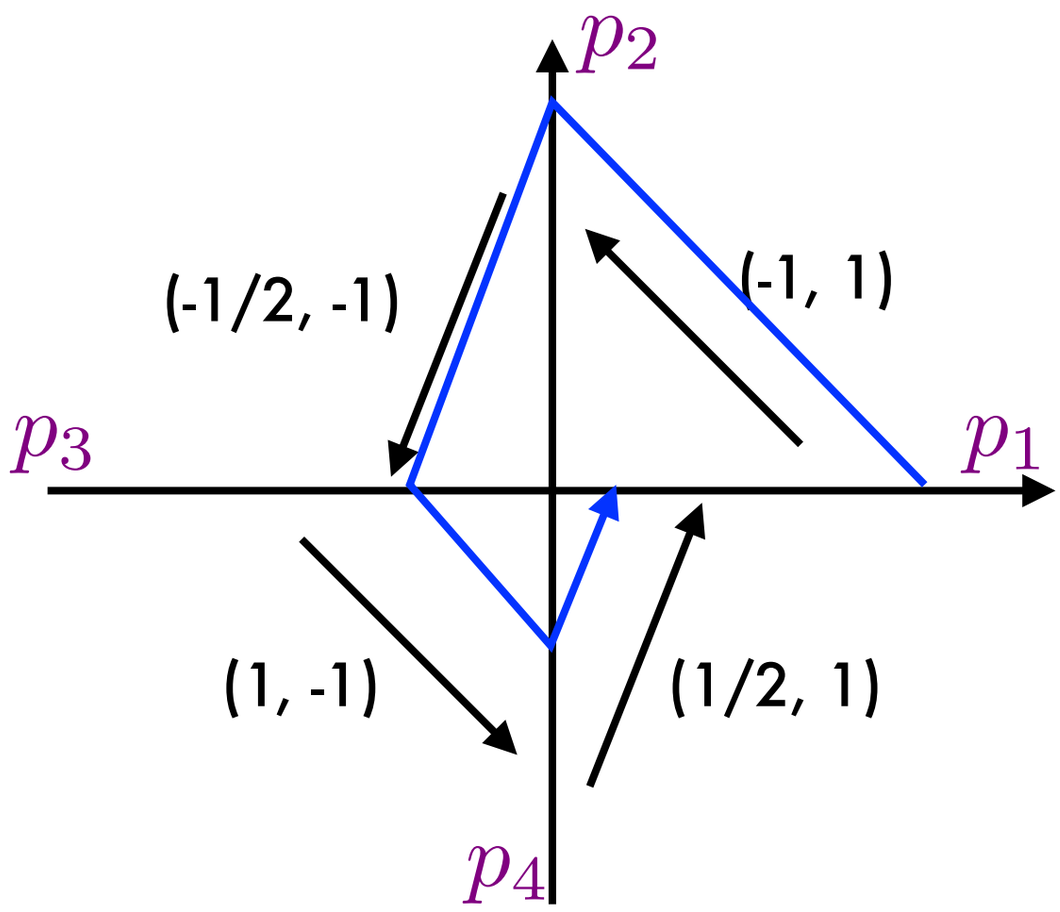


Lyapunov stable, but not asymptotically stable

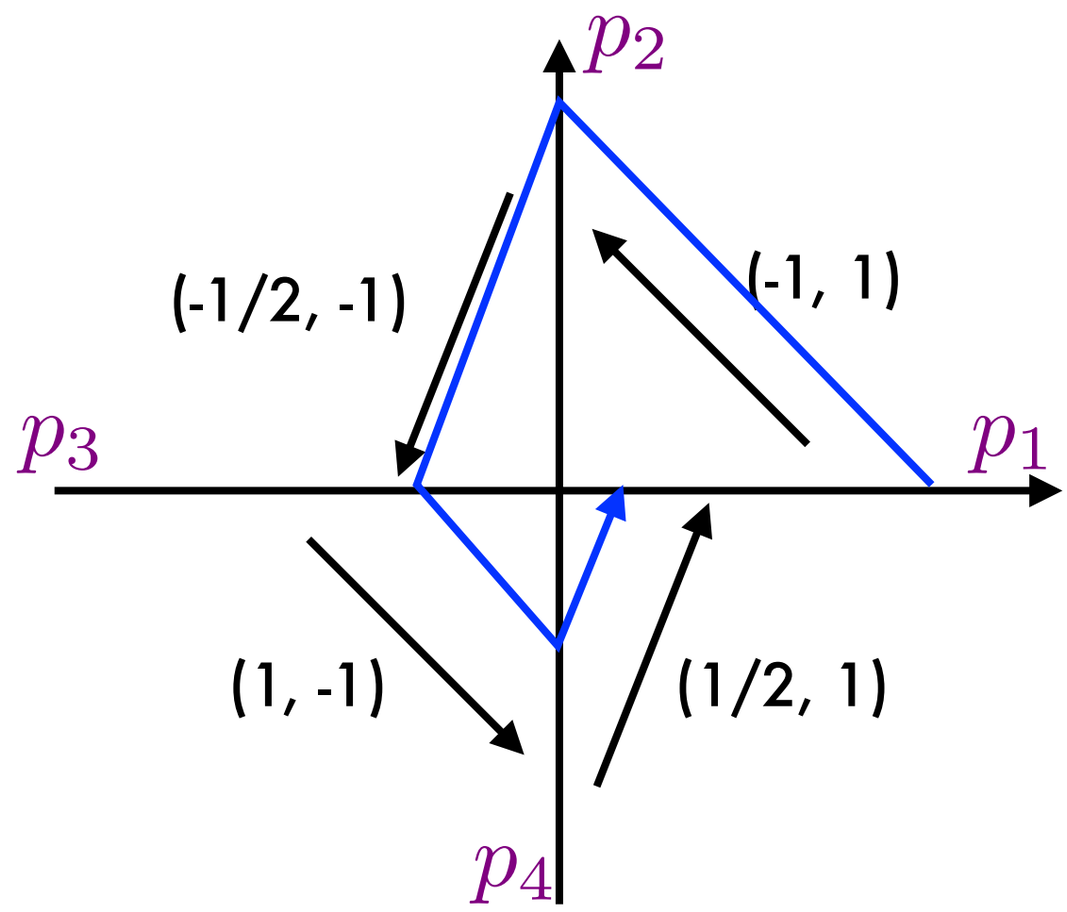
Piecewise Constant Derivative System



Piecewise Constant Derivative System

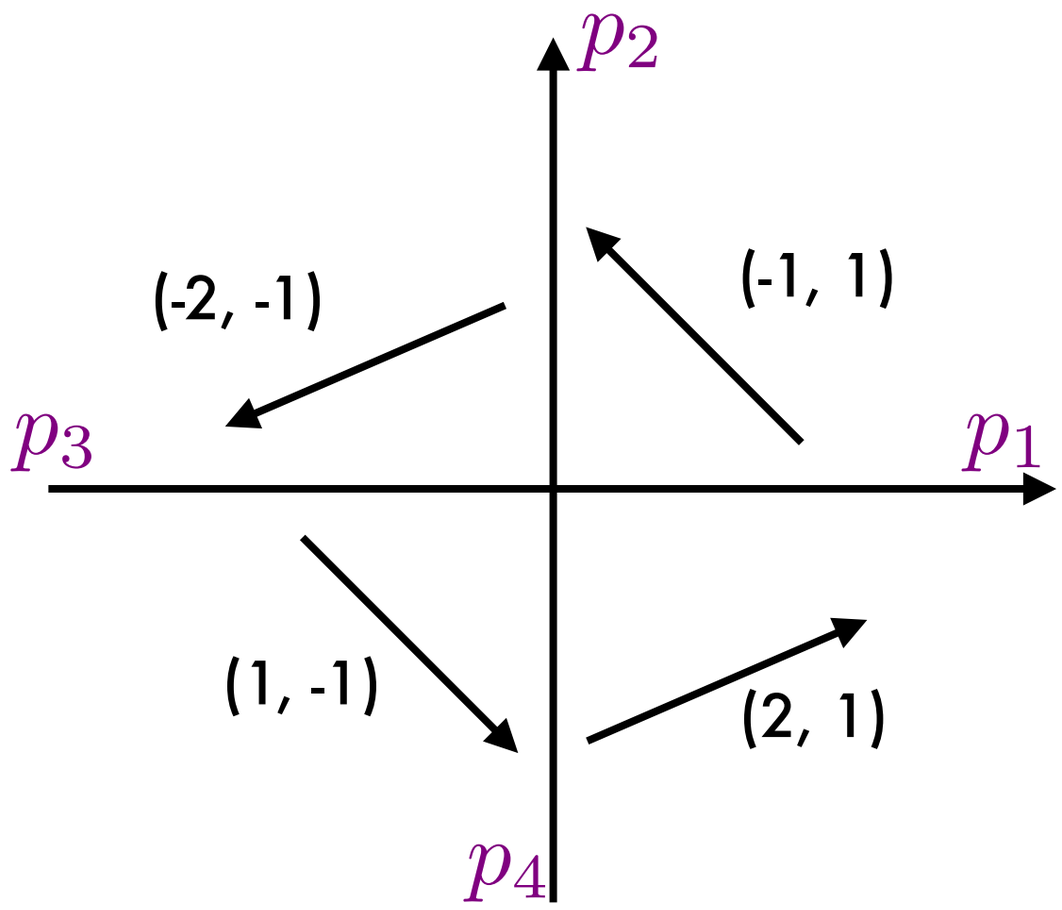


Piecewise Constant Derivative System

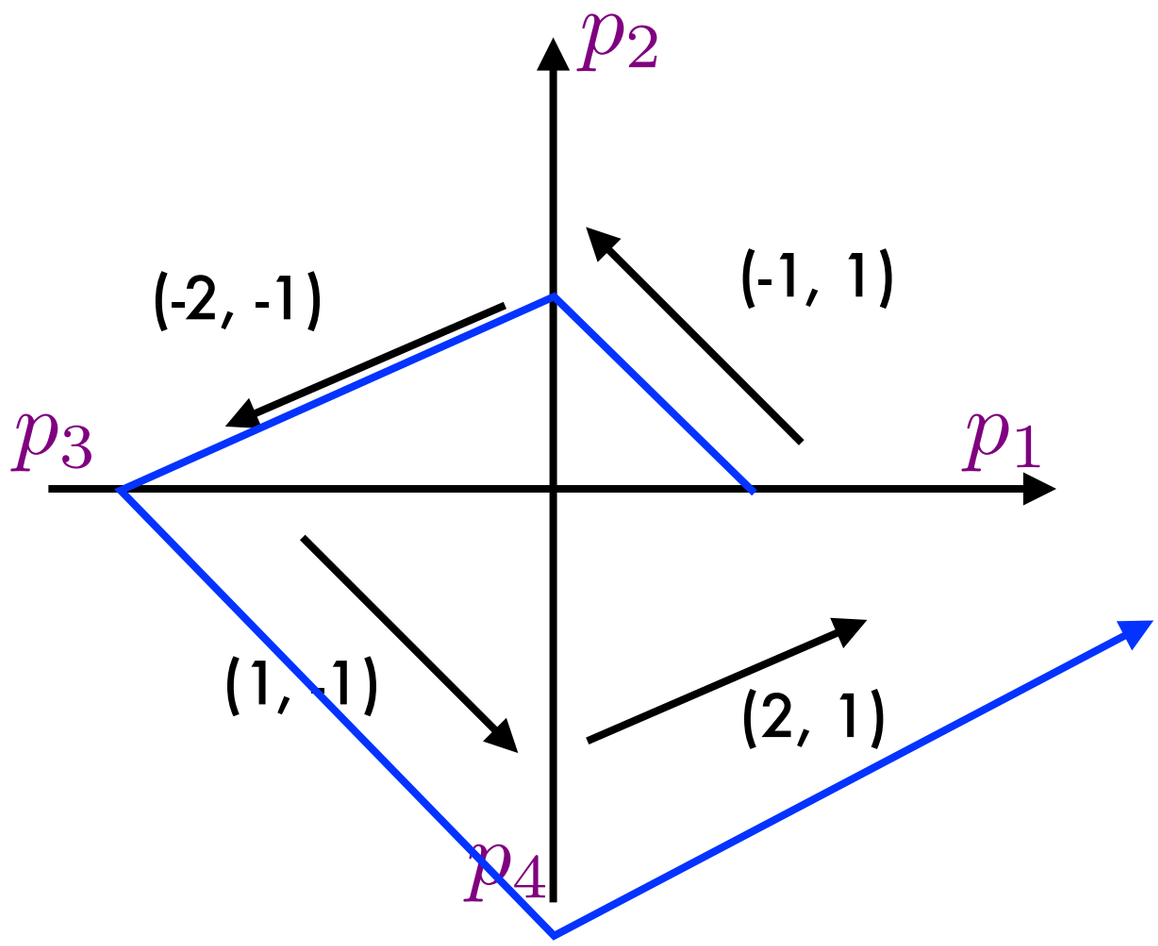


Lyapunov stable and asymptotically stable

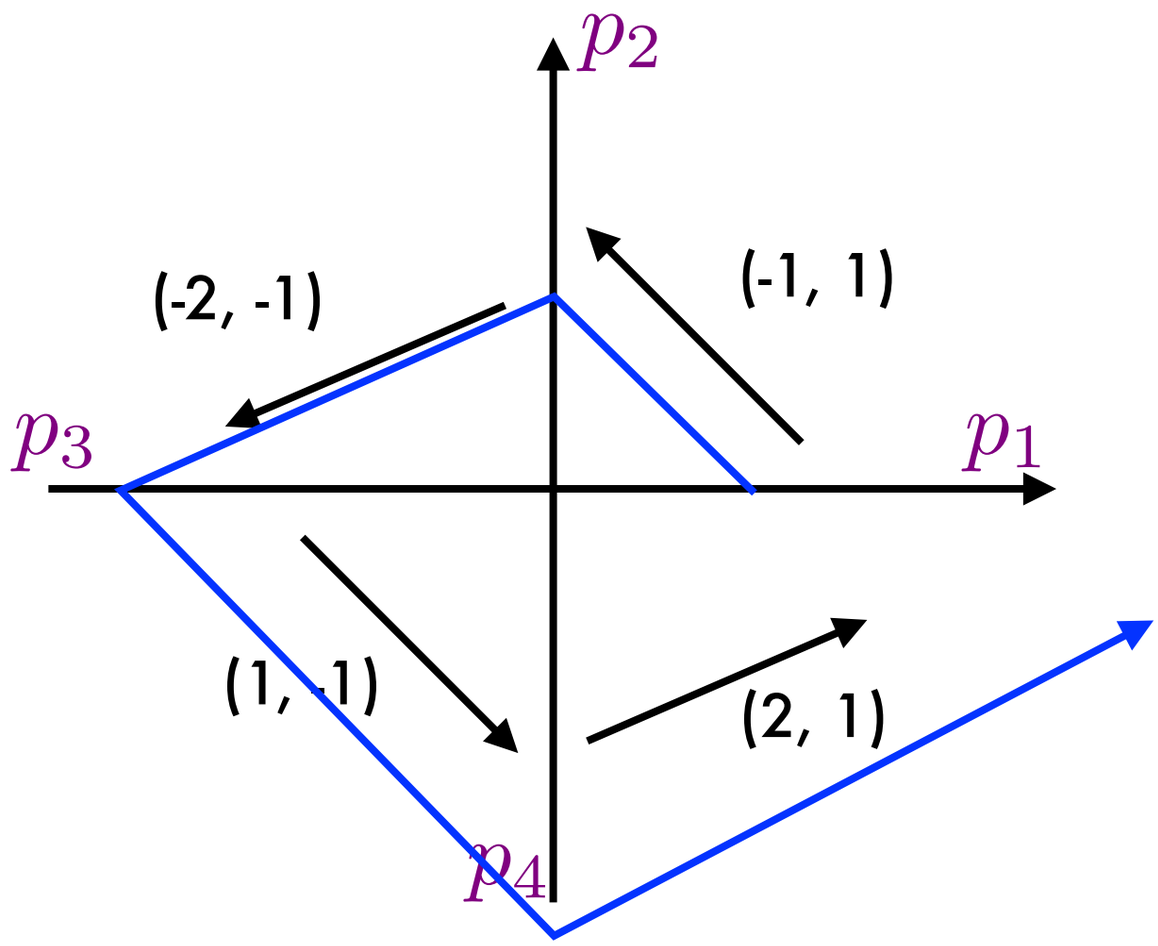
Piecewise Constant Derivative System



Piecewise Constant Derivative System

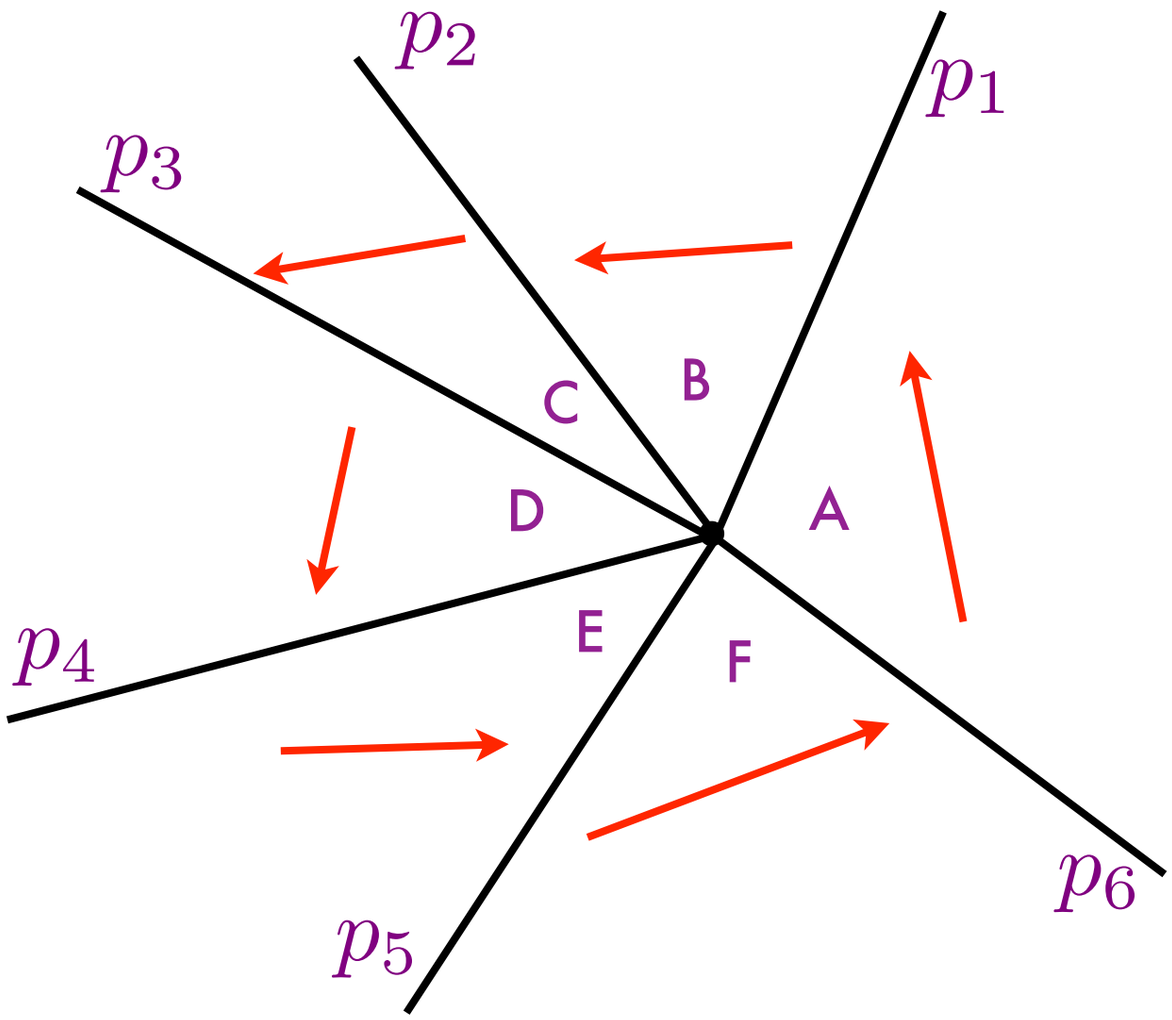


Piecewise Constant Derivative System

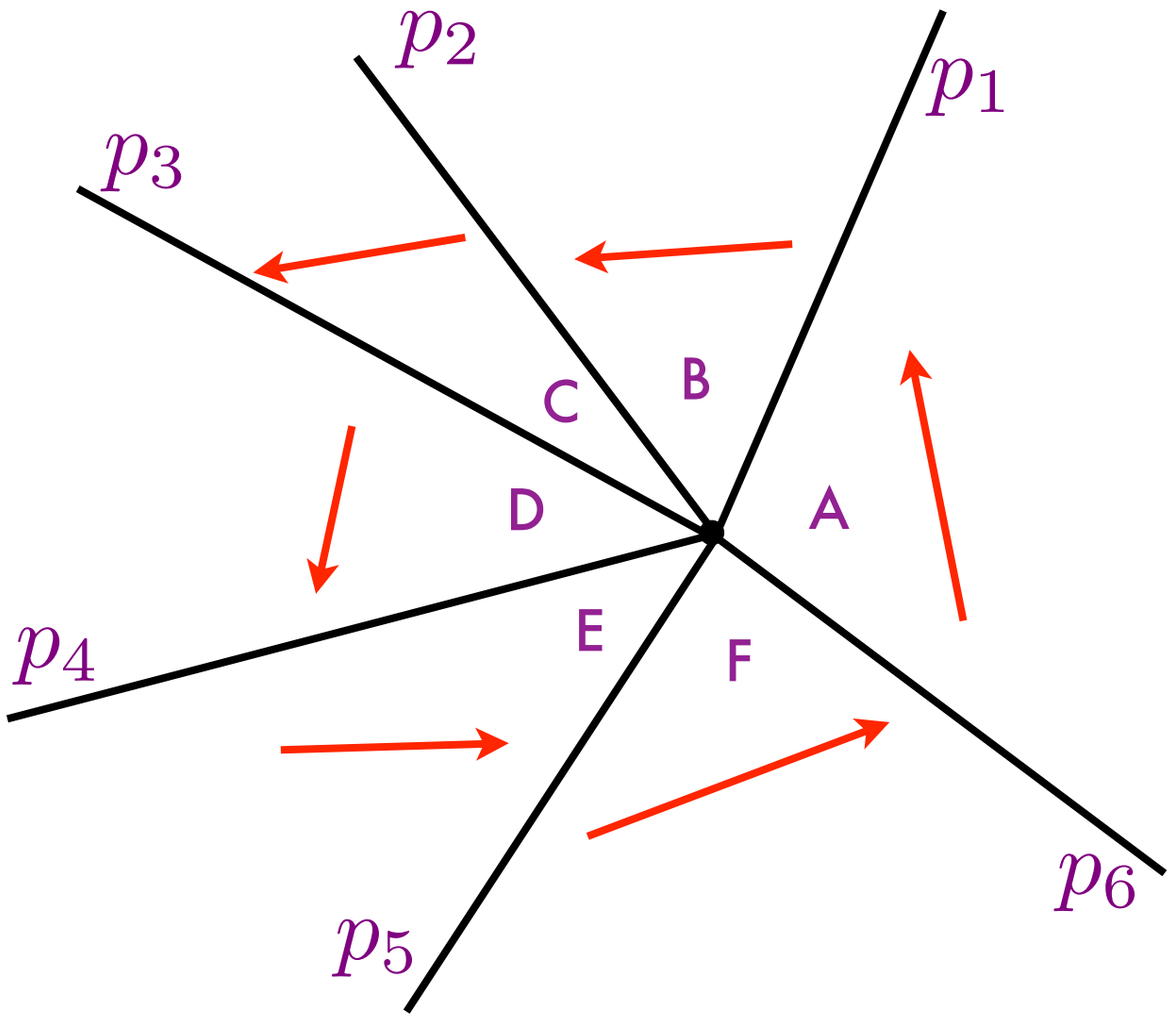


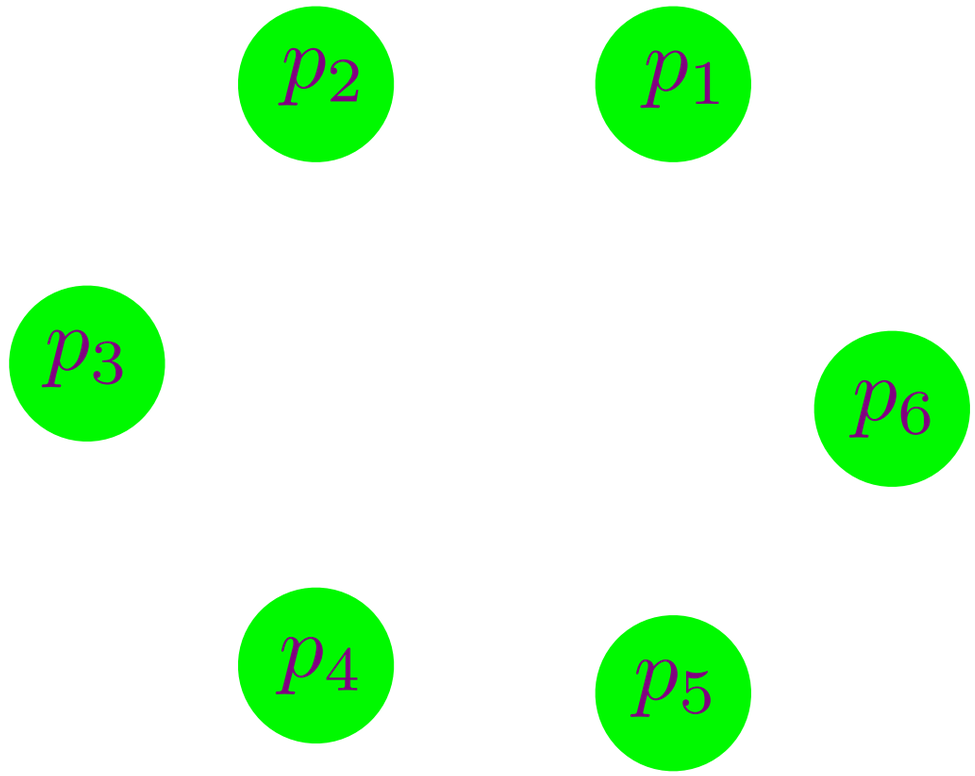
Neither Lyapunov stable nor asymptotically stable

Stability Analysis: Graph Construction

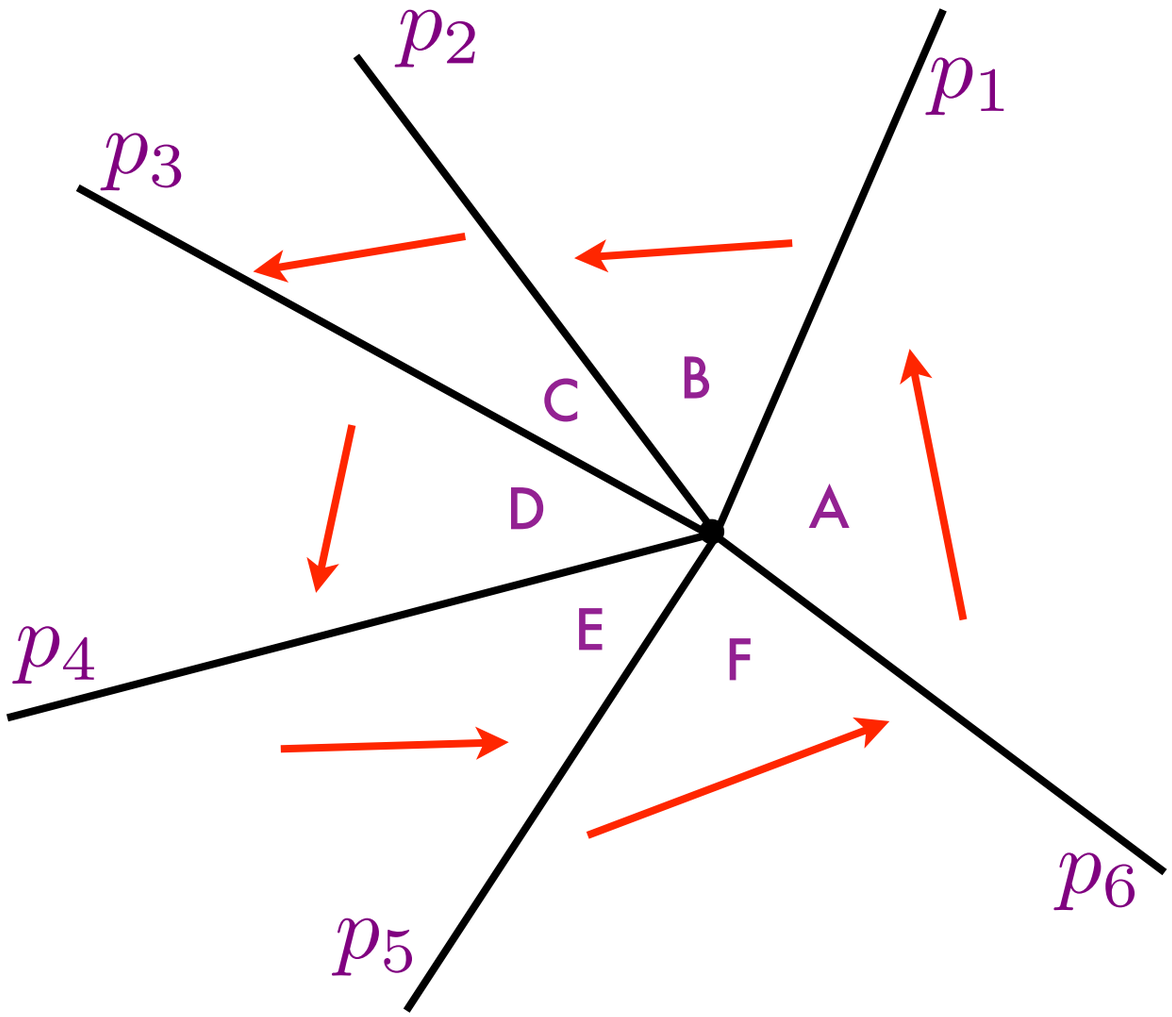


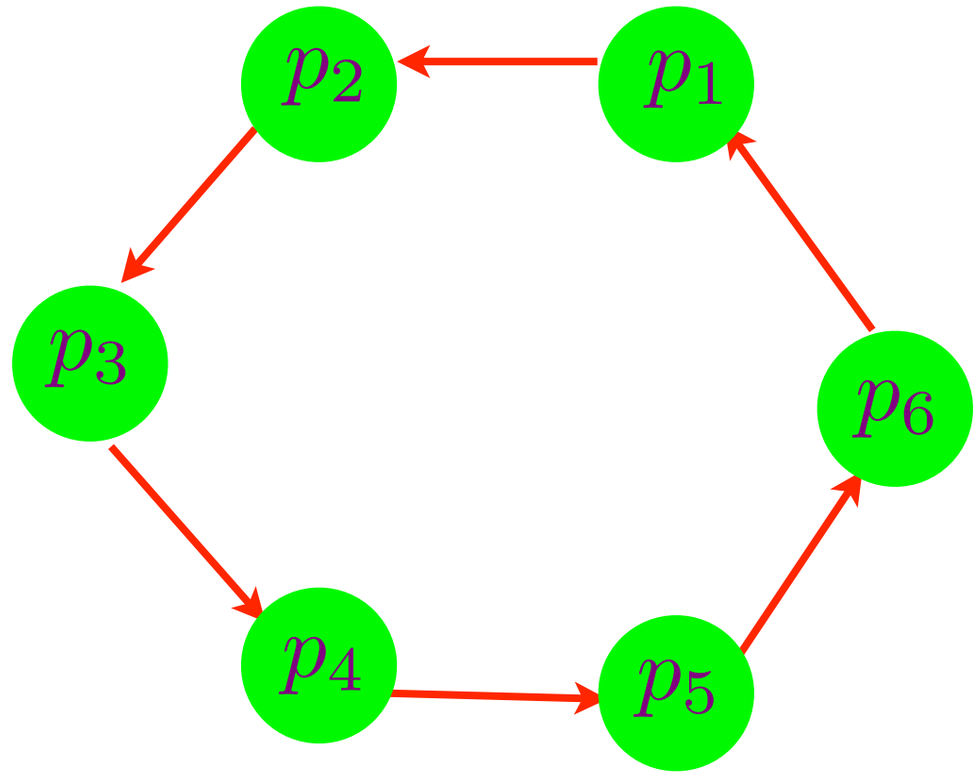
Stability Analysis: Graph Construction



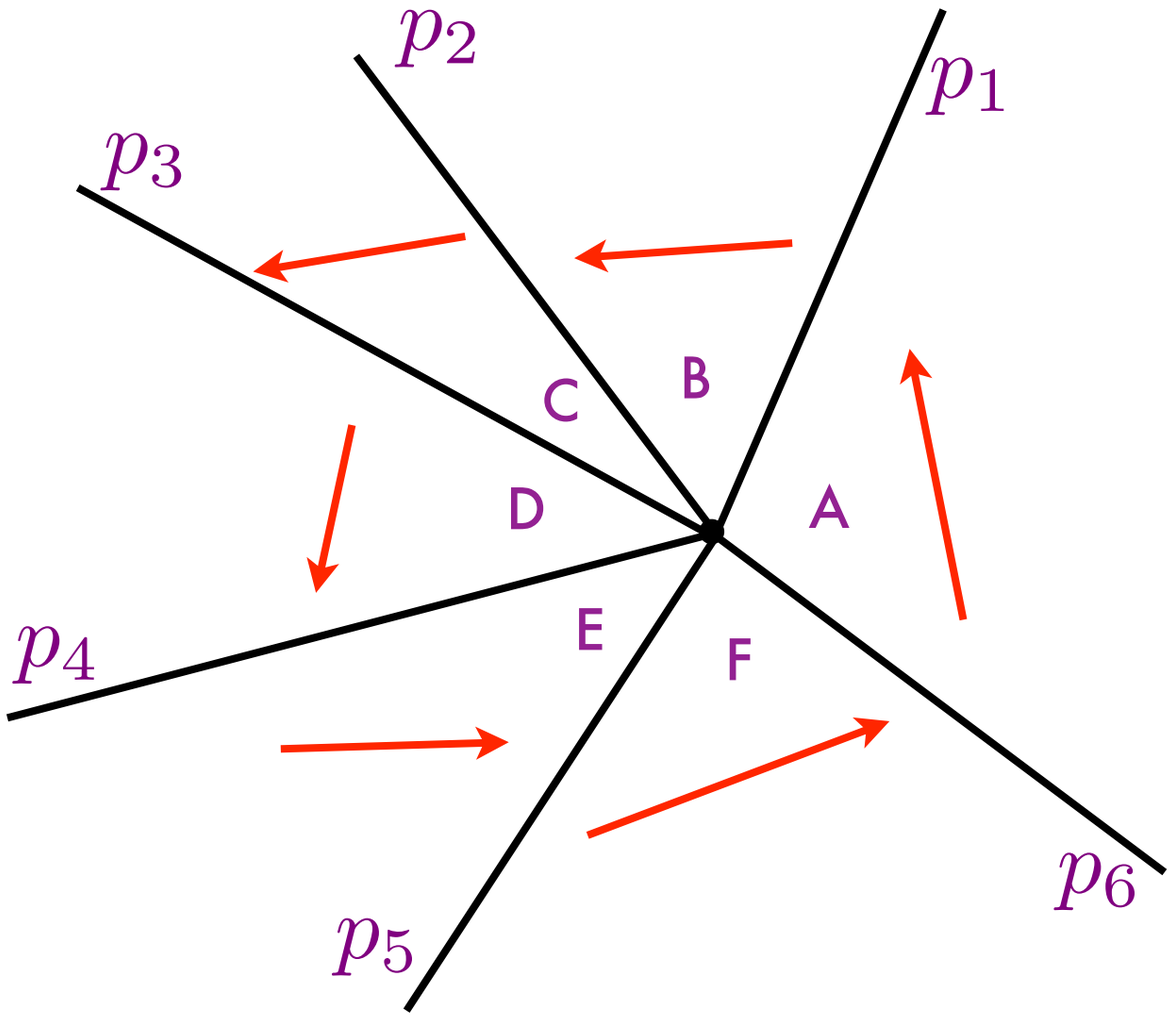


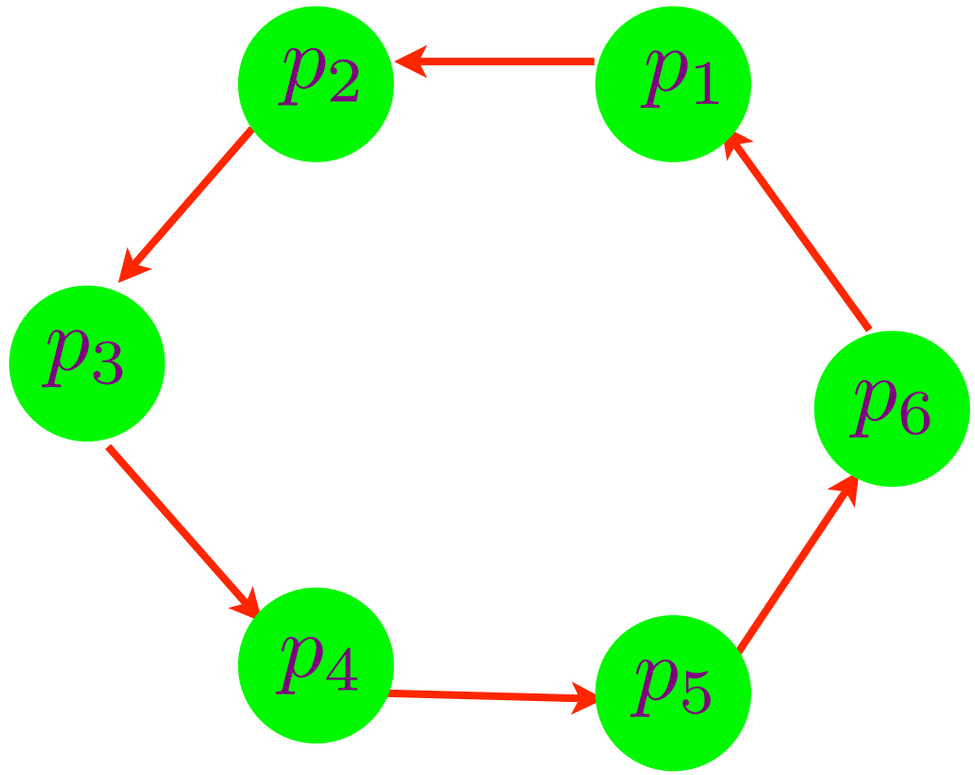
Stability Analysis: Graph Construction





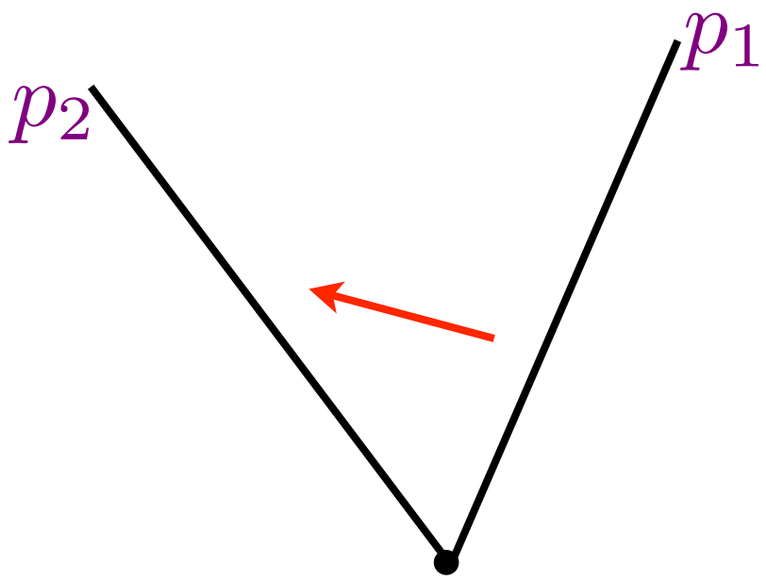
Stability Analysis: Graph Construction





Capture information about distance to the origin along the executions

Stability Analysis: Weight computation



Need to capture information about distance to the origin along the executions

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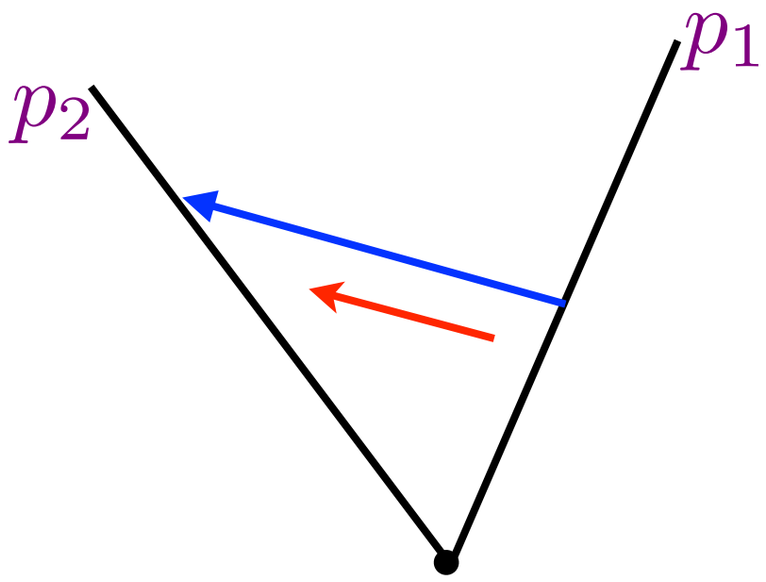

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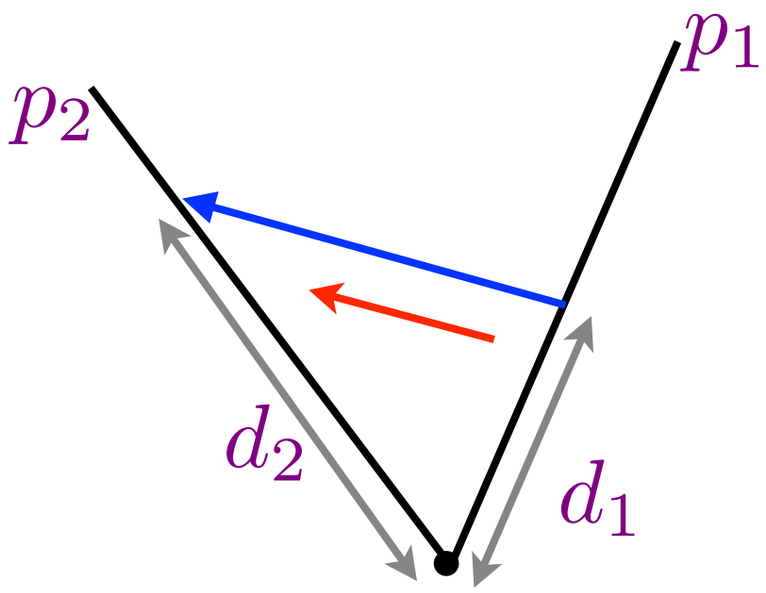

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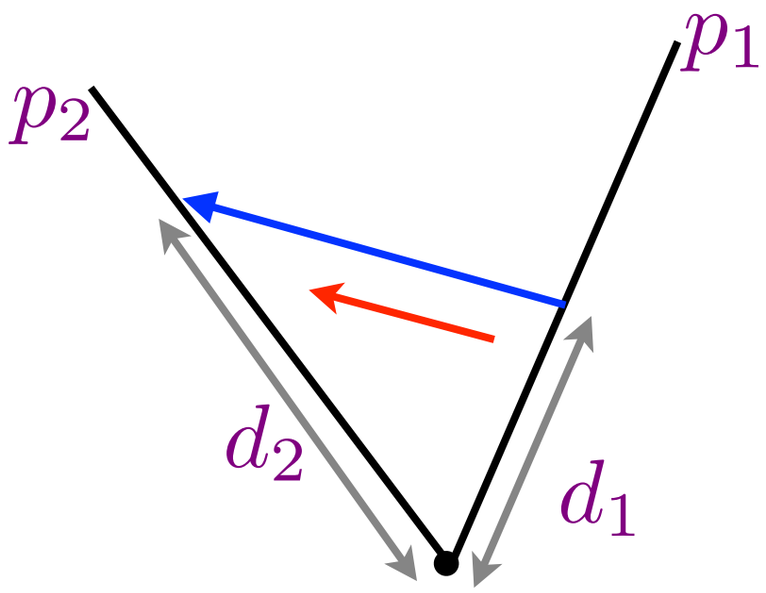
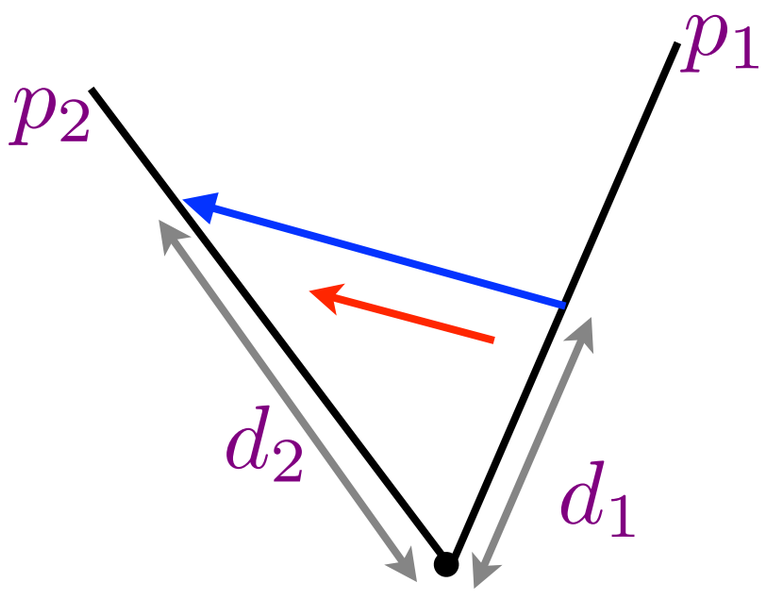


$$w(e) = \frac{|d_2|}{|d_1|}$$

Need to capture information about distance to the origin along the executions

Weight computation



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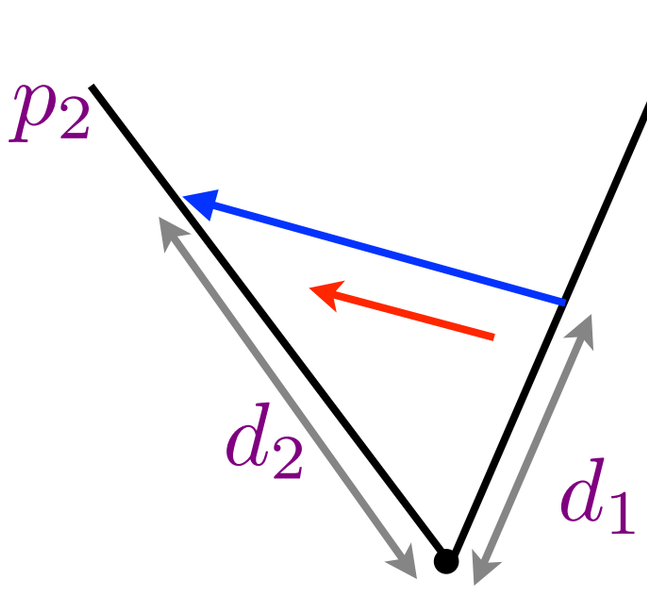

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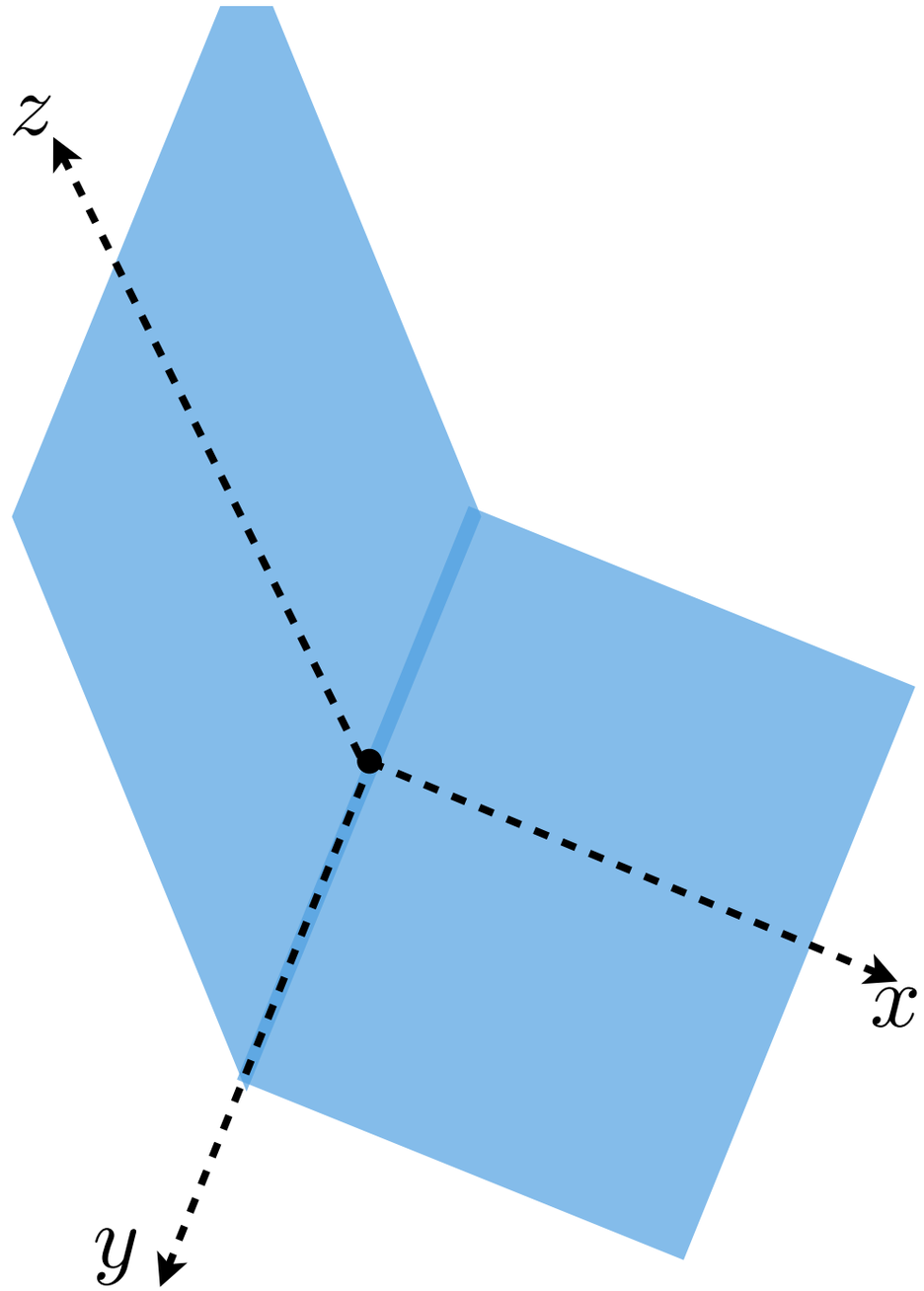
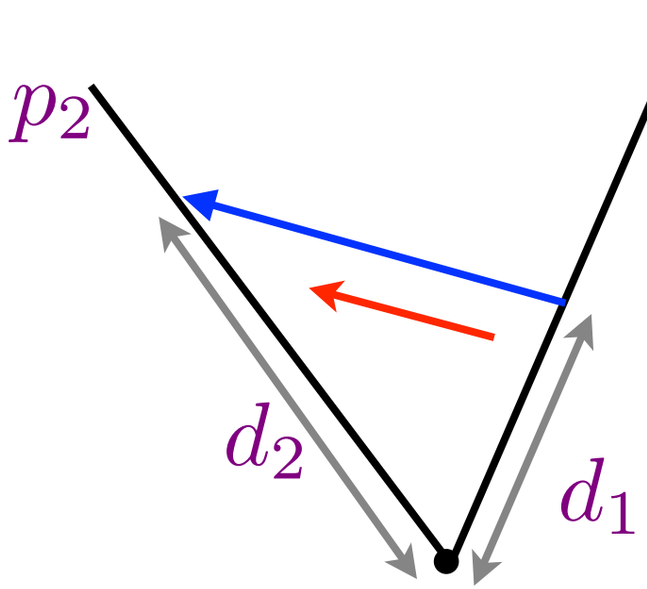


Weight computation



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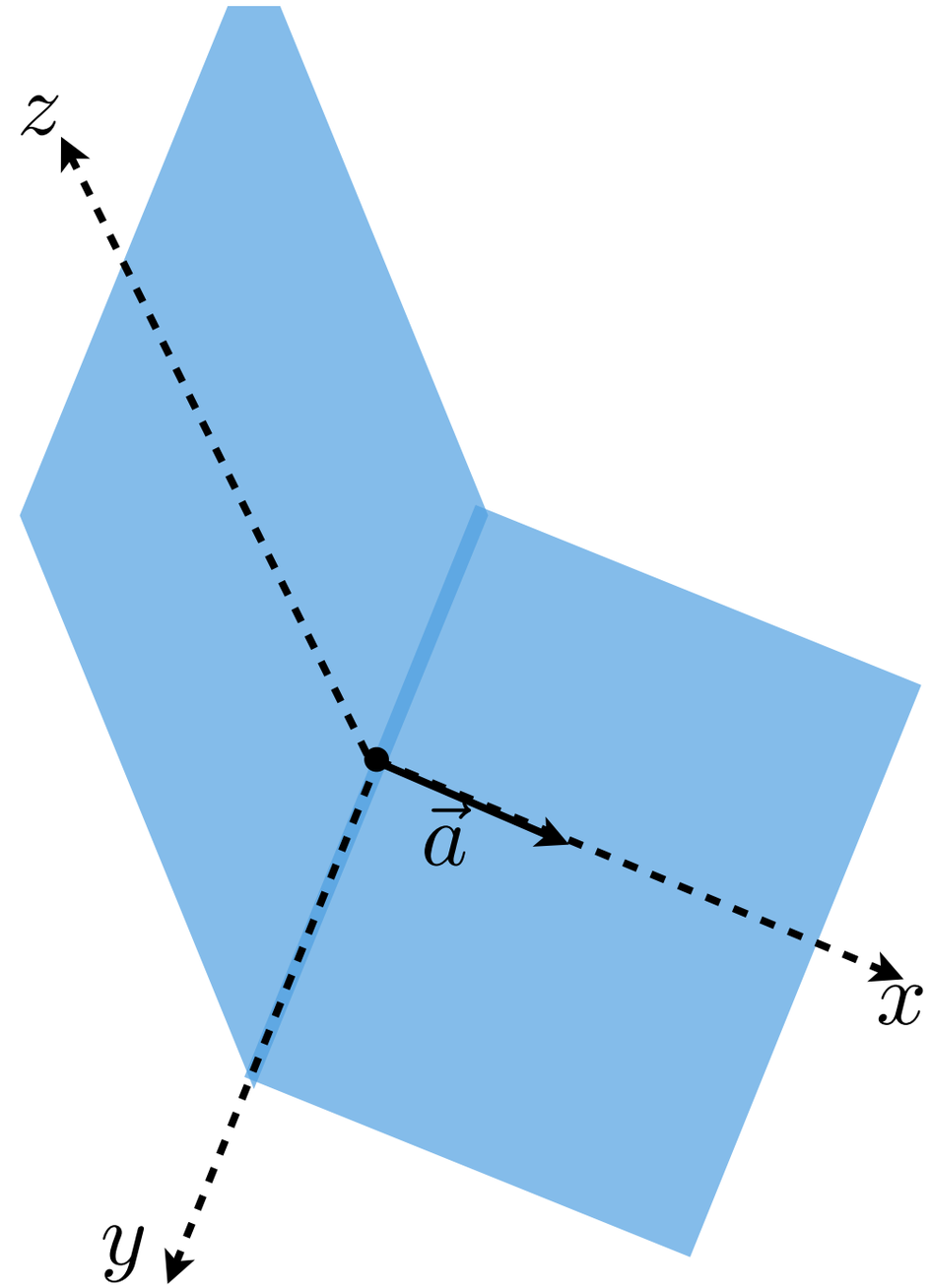
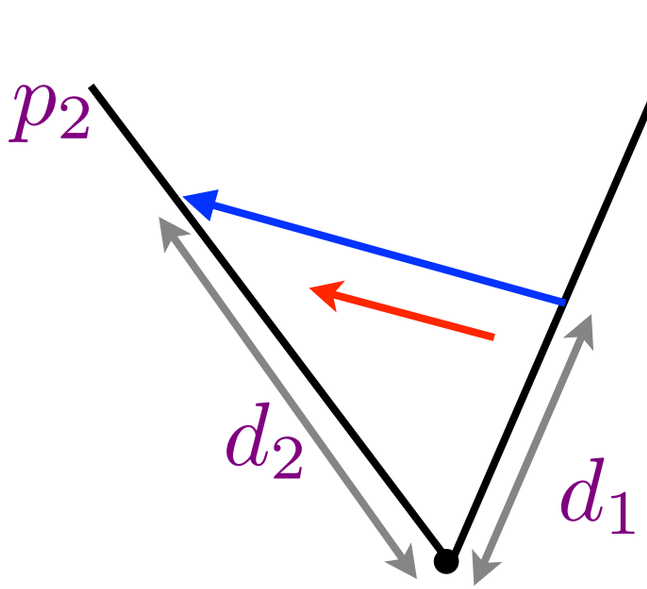


Weight computation



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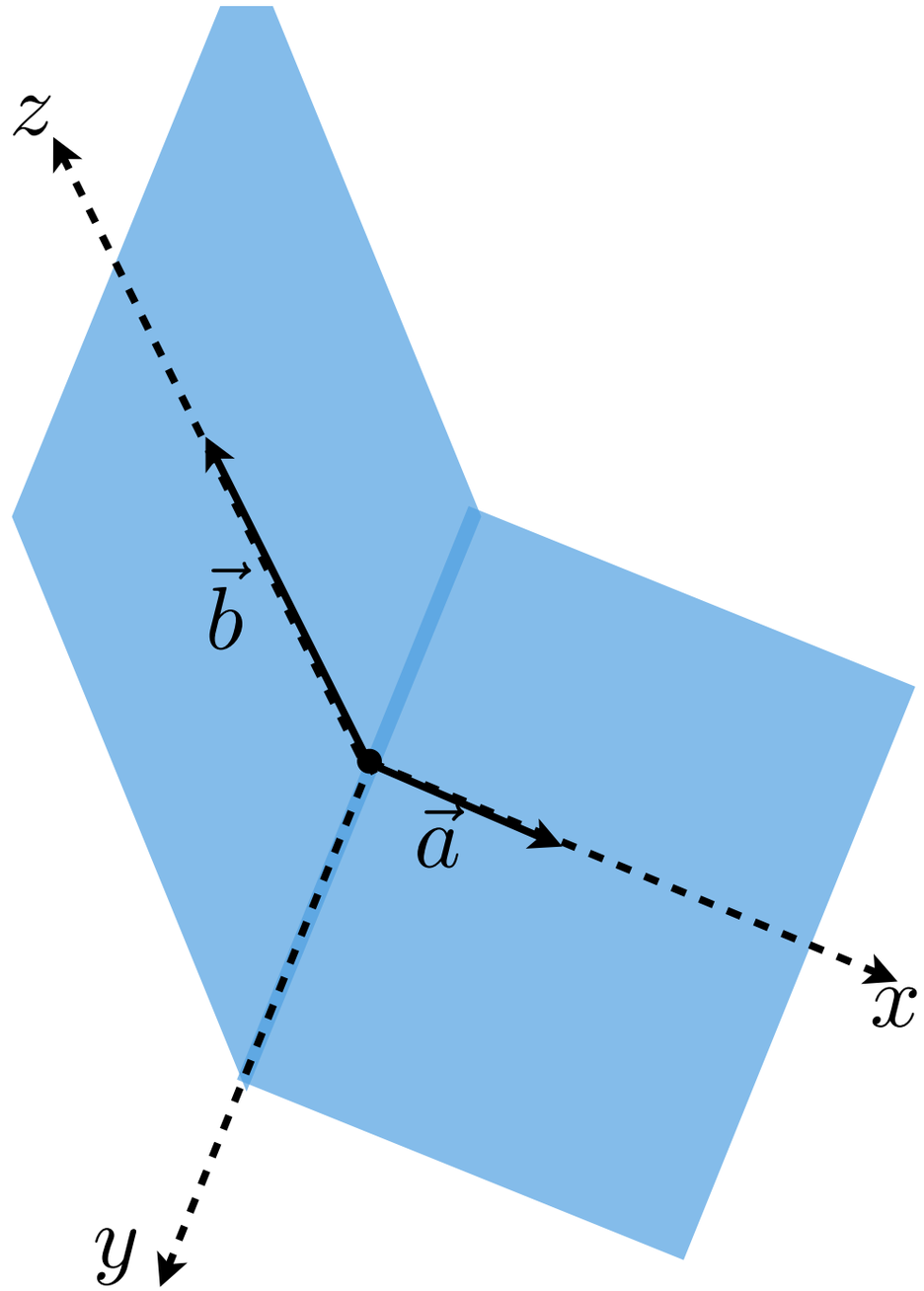
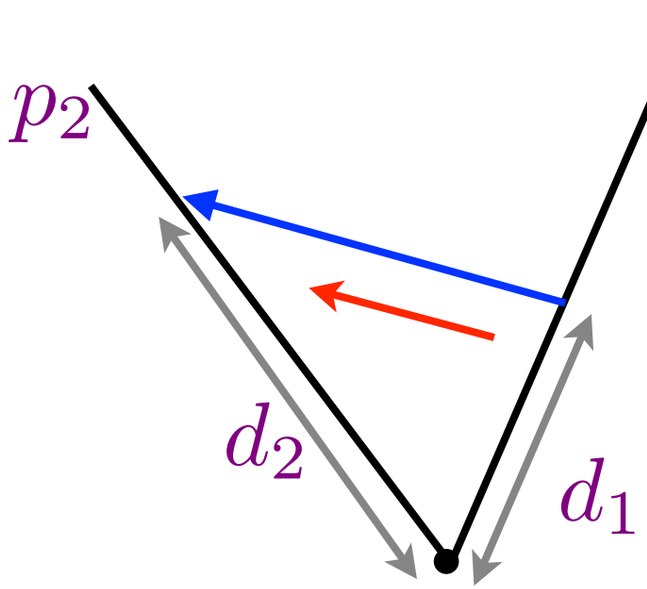


Weight computation



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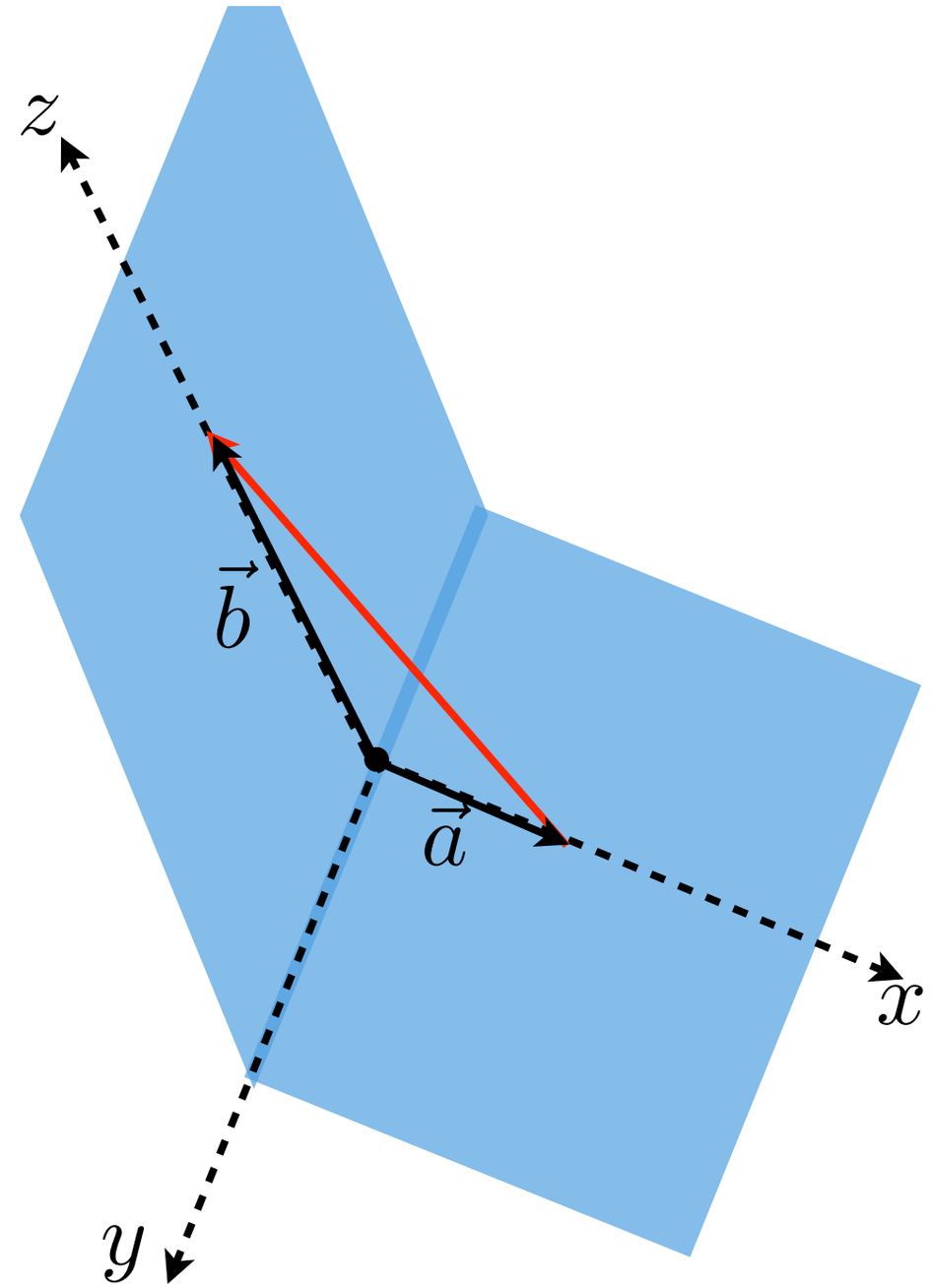
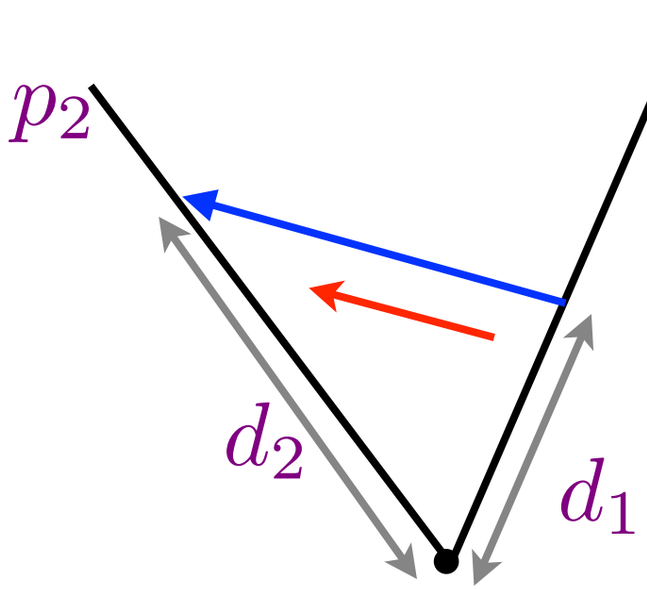


Weight computation



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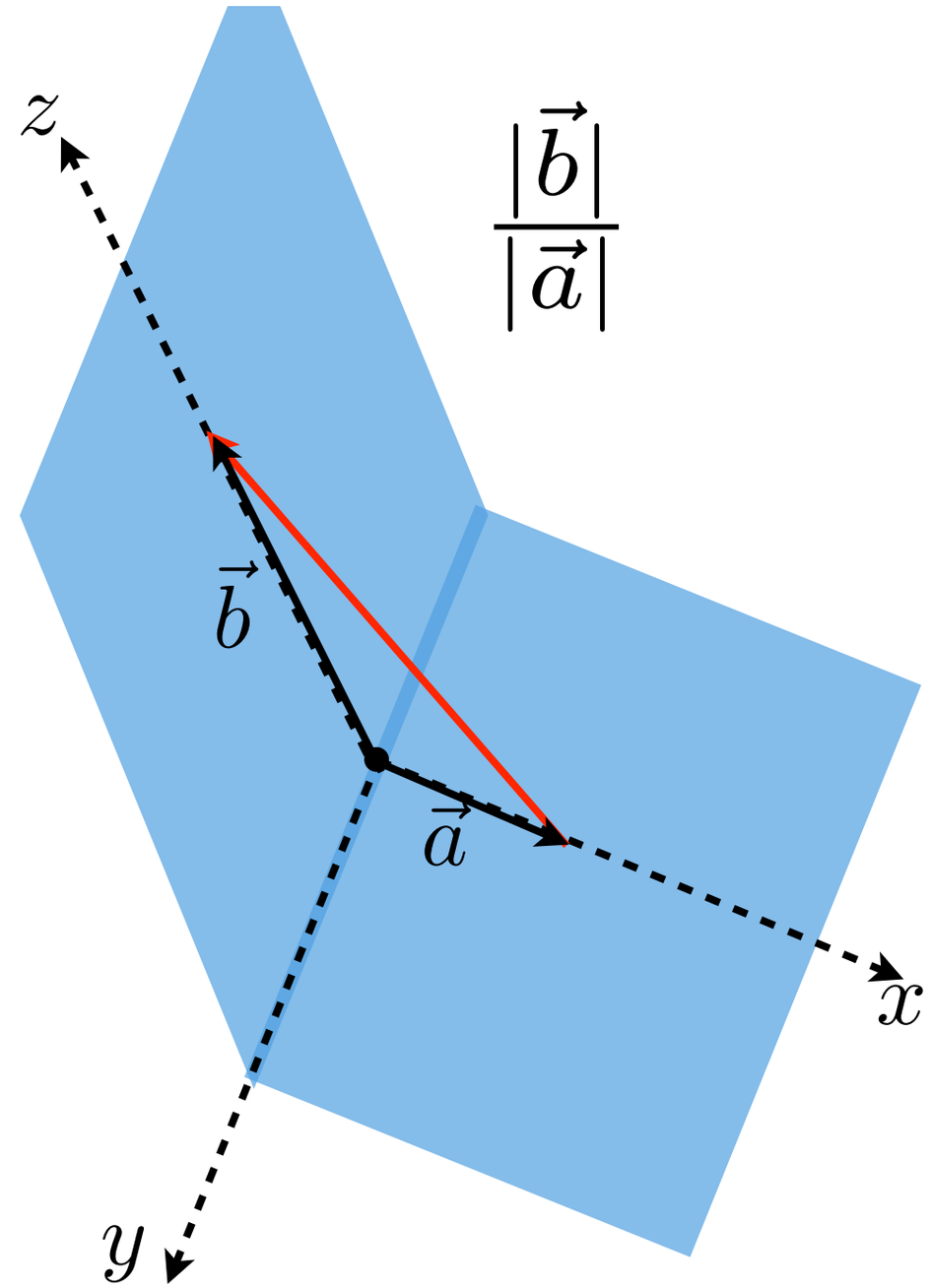
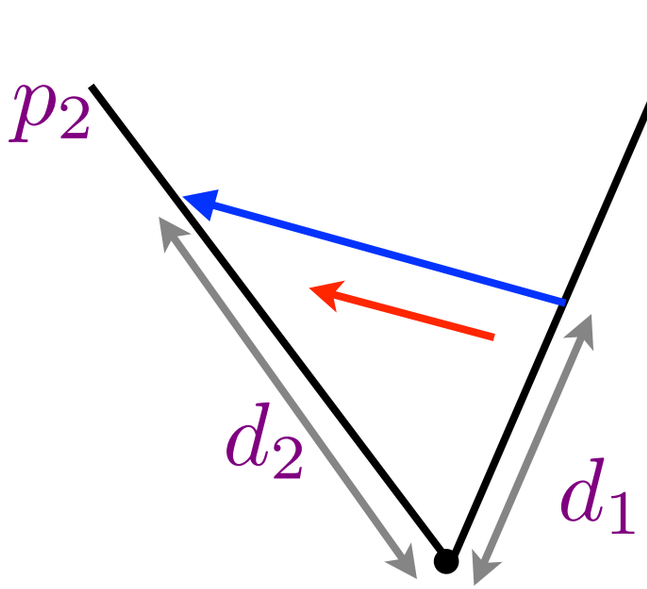


Weight computation



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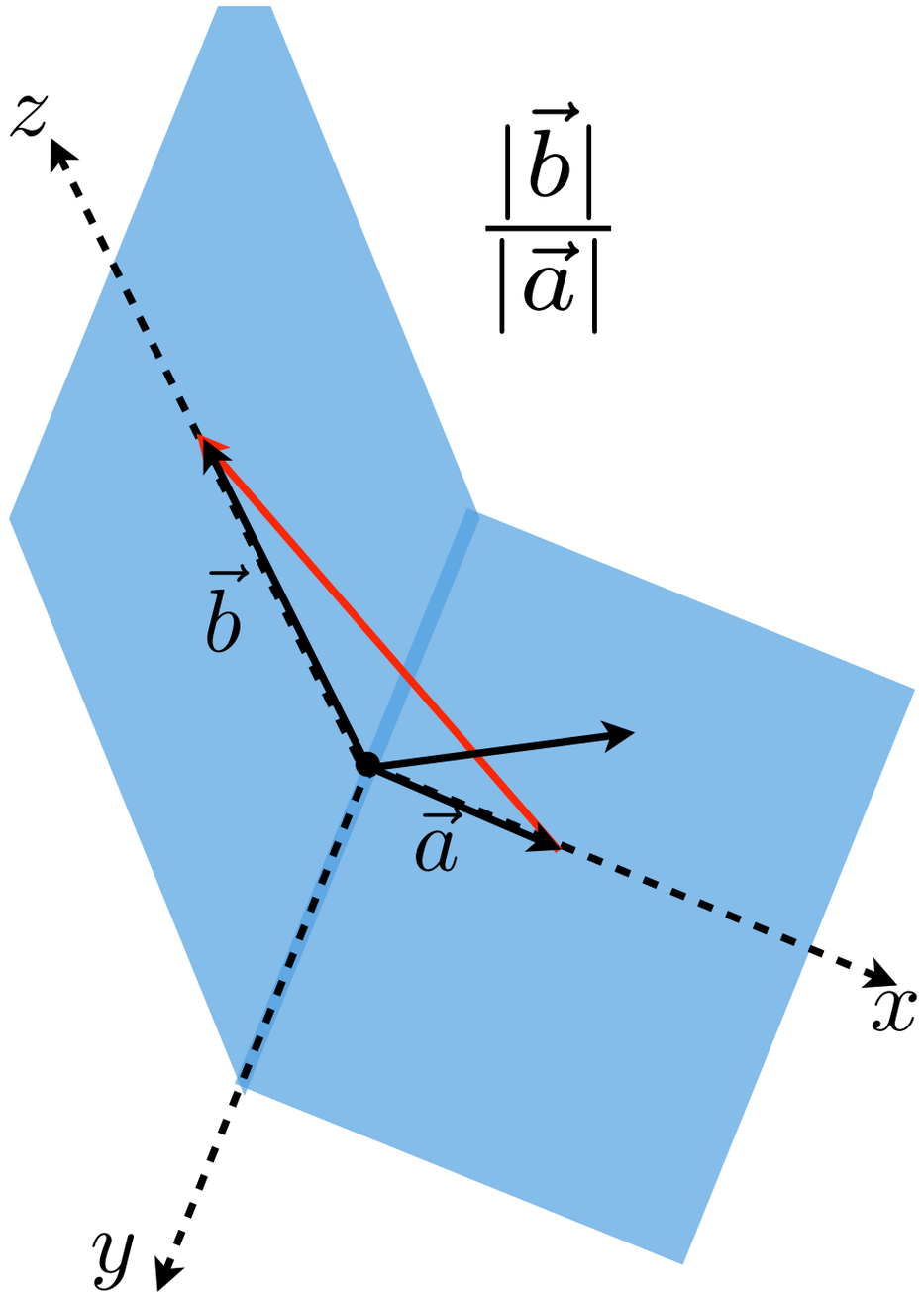
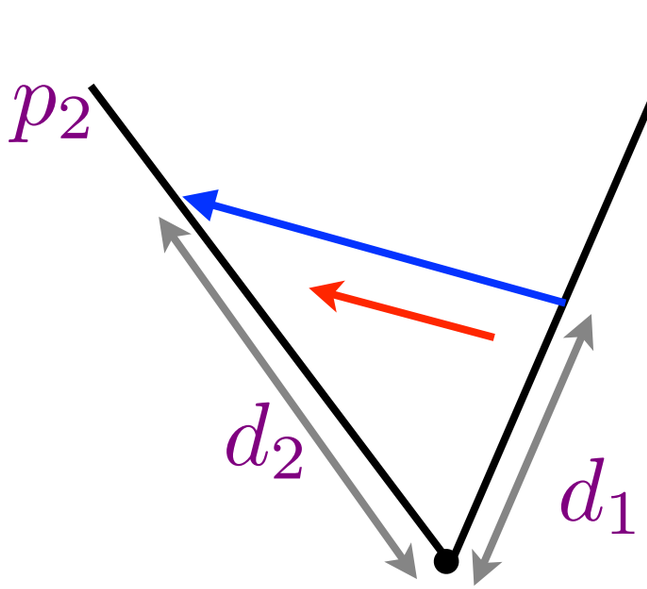


Weight computation



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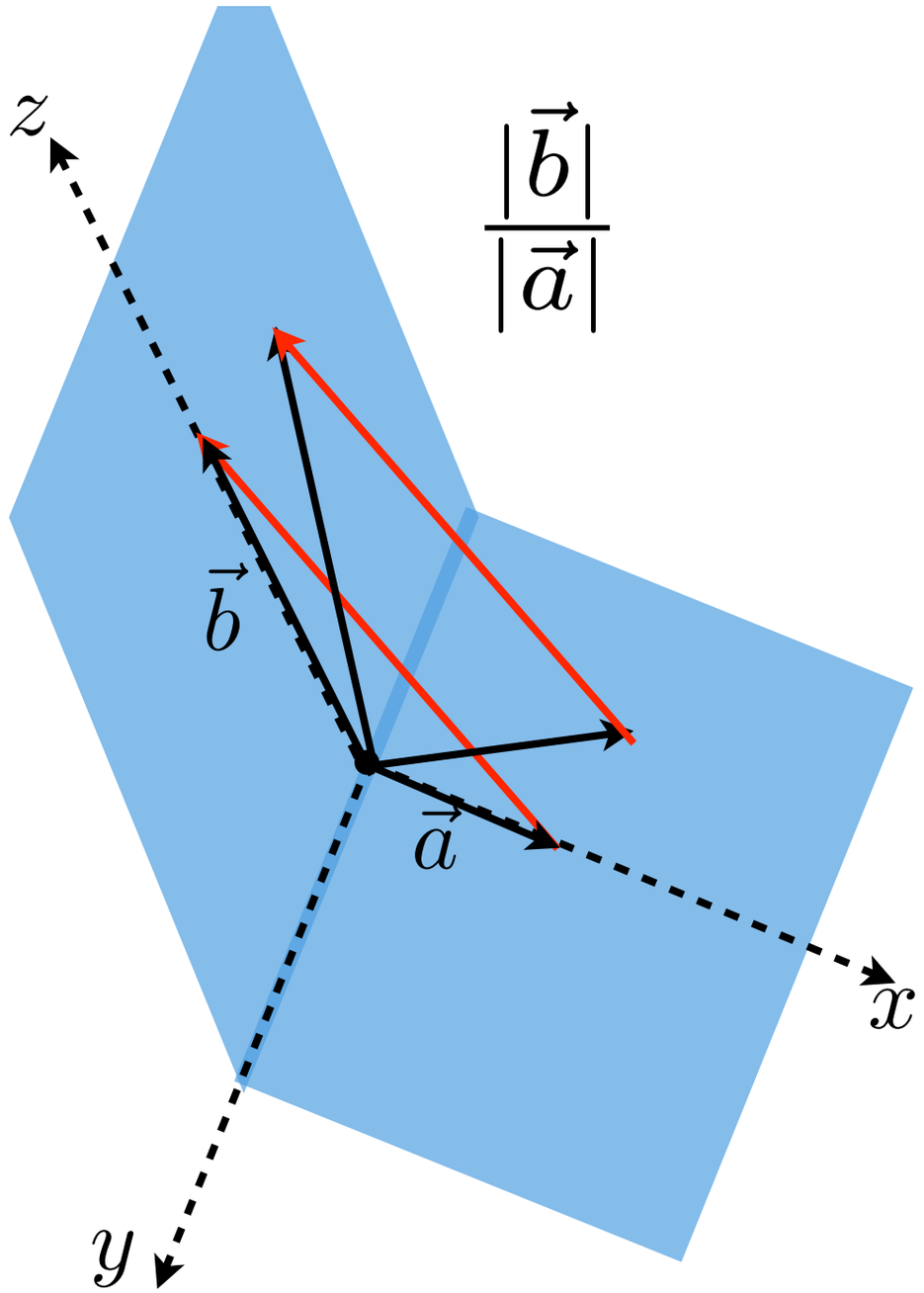
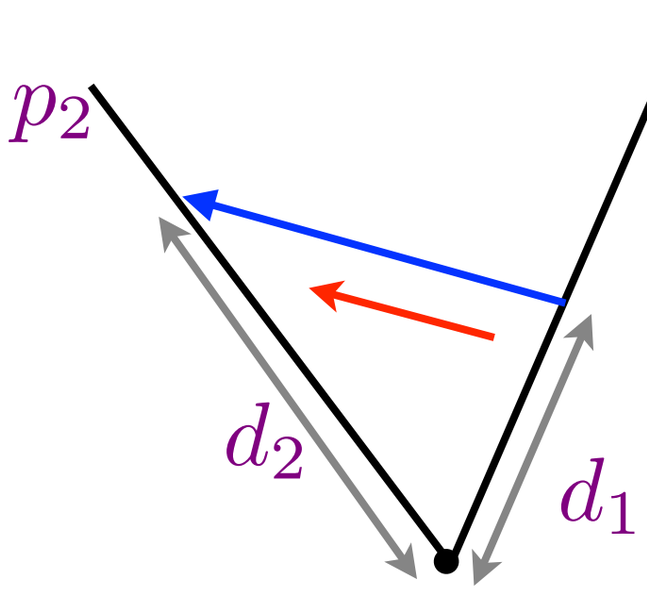


Weight computation



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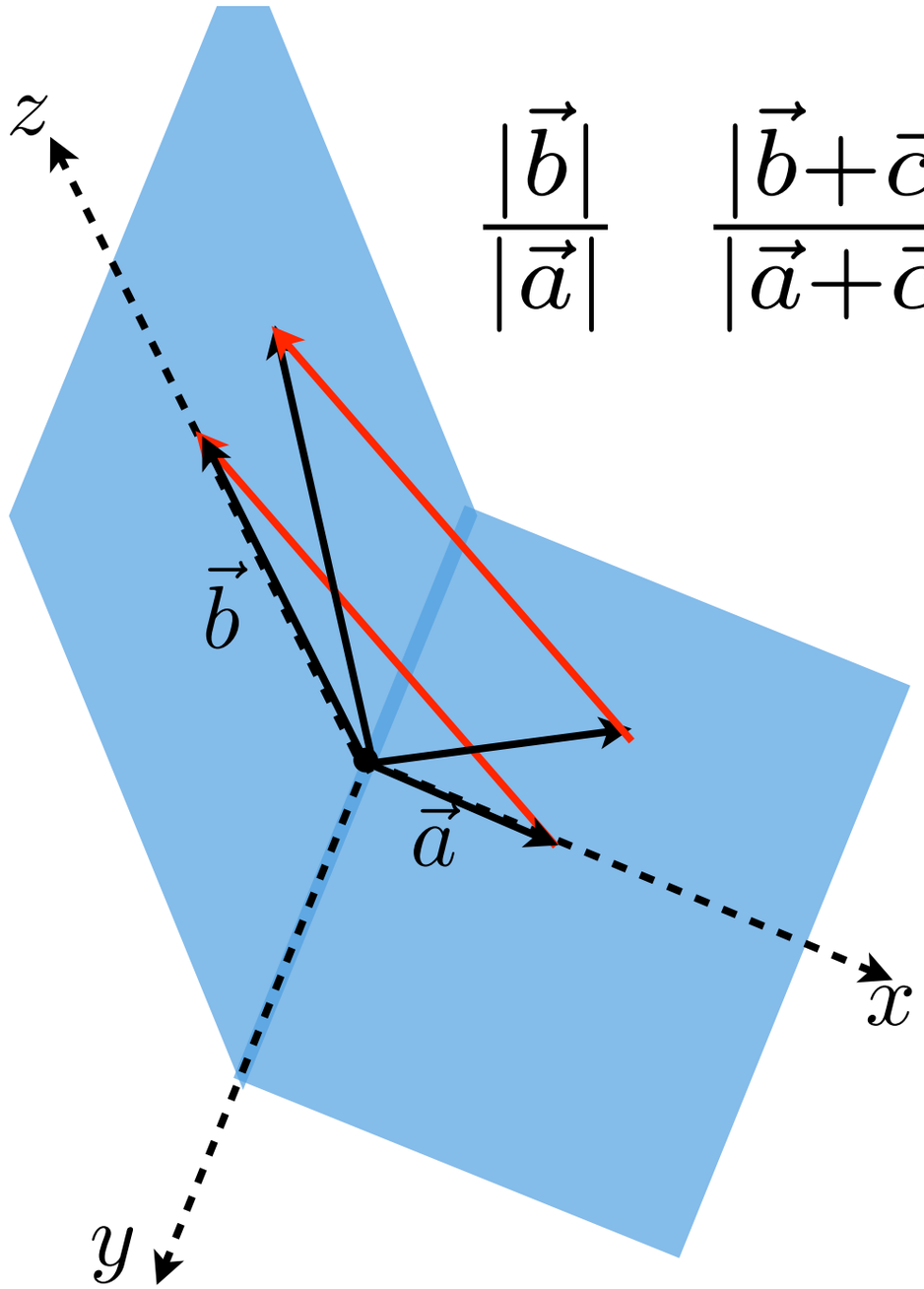
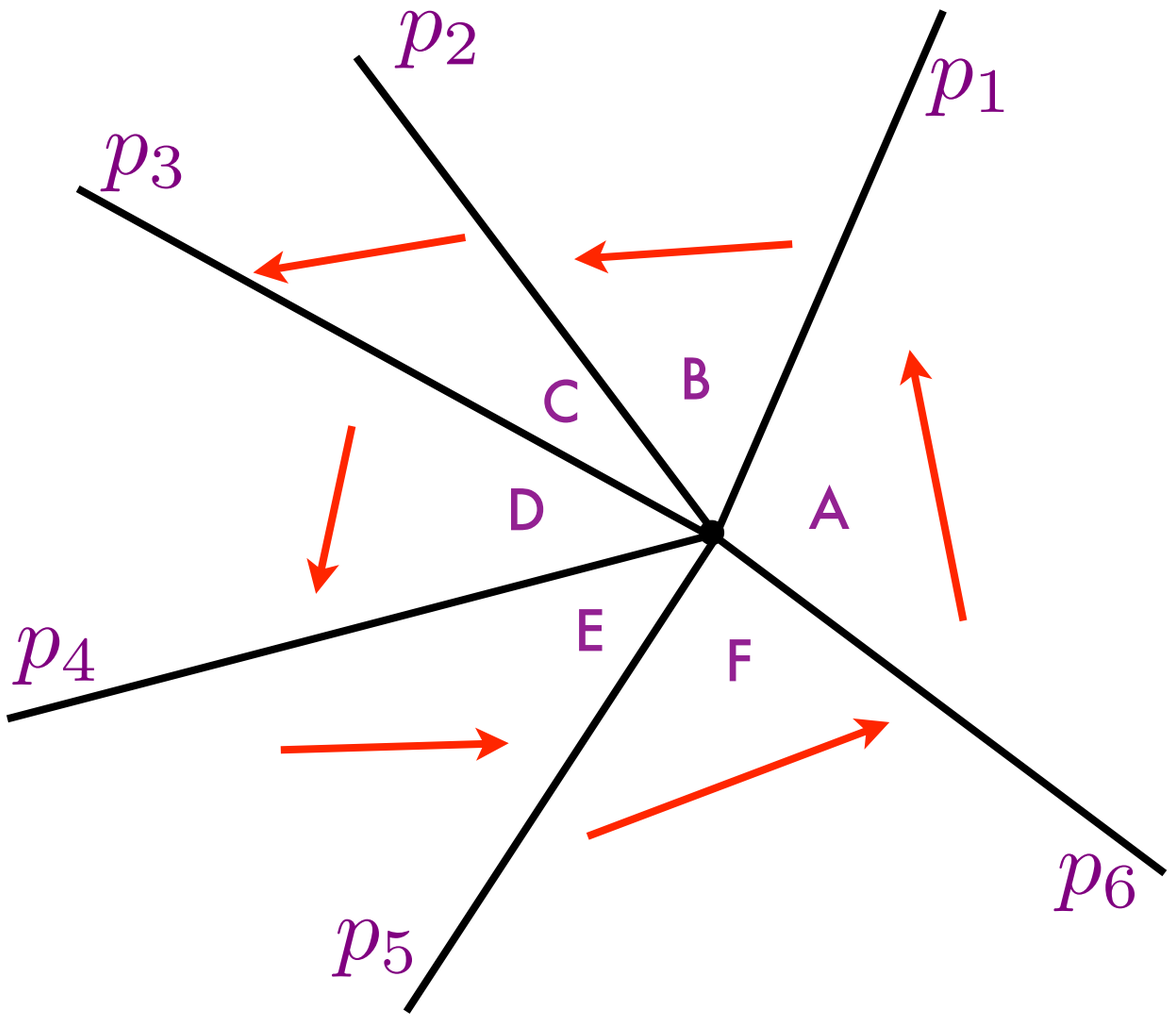
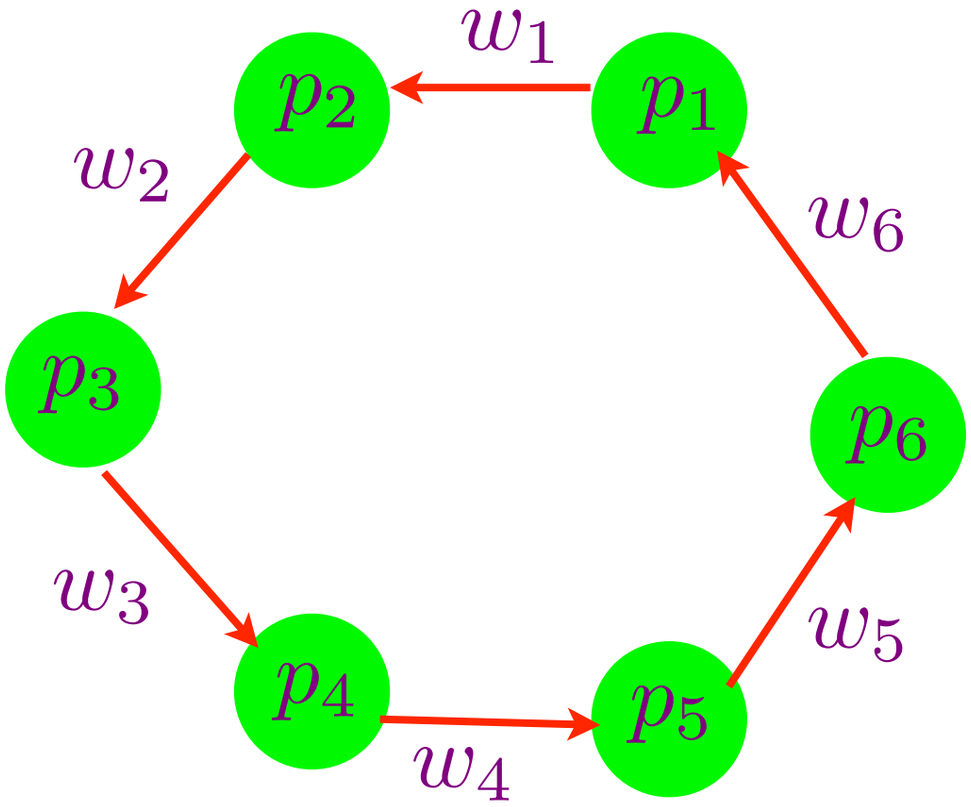


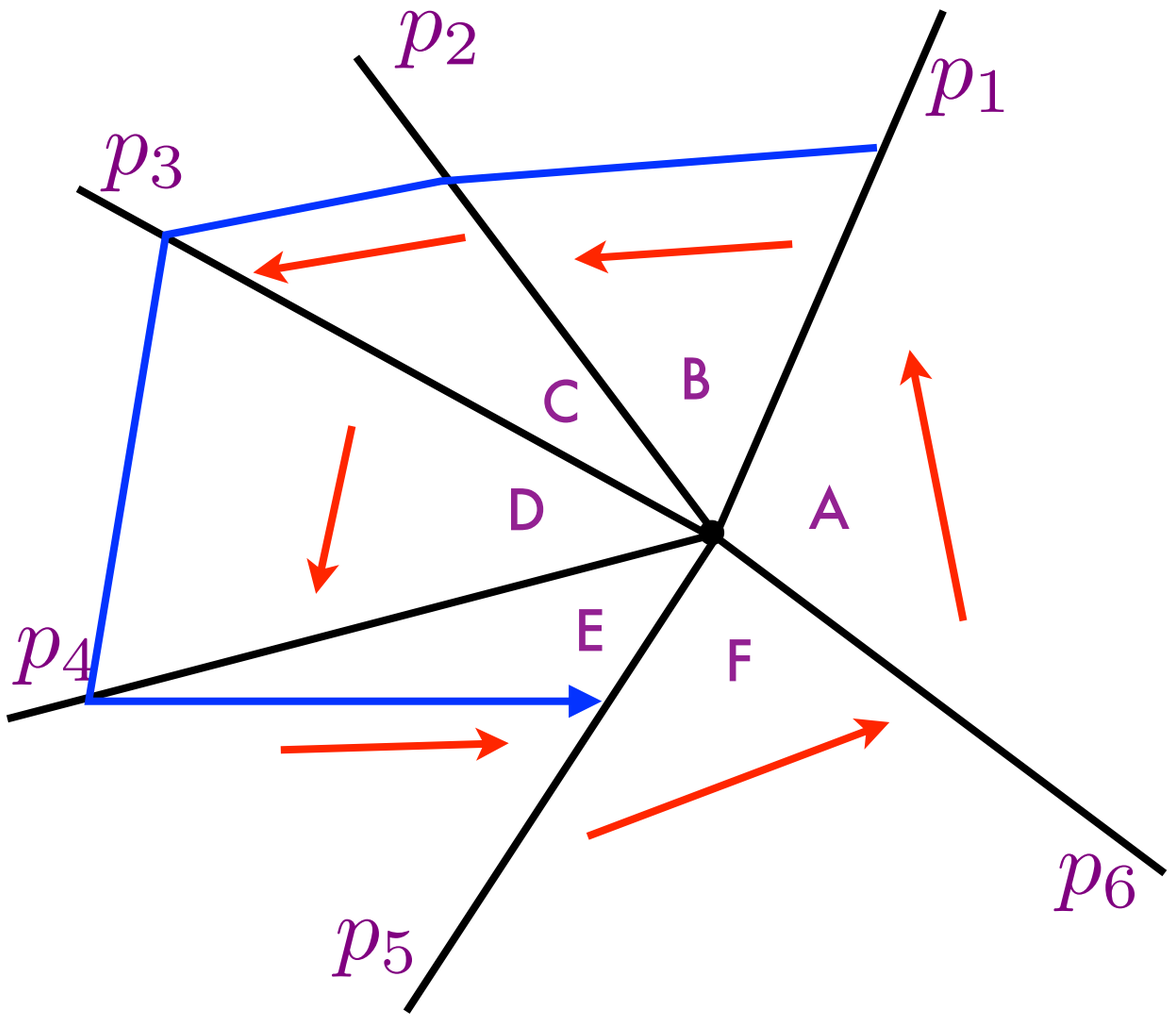
$$\frac{|\vec{b}|}{|\vec{a}|} \qquad \frac{|\vec{b}+\vec{c}|}{|\vec{a}+\vec{c}|}$$

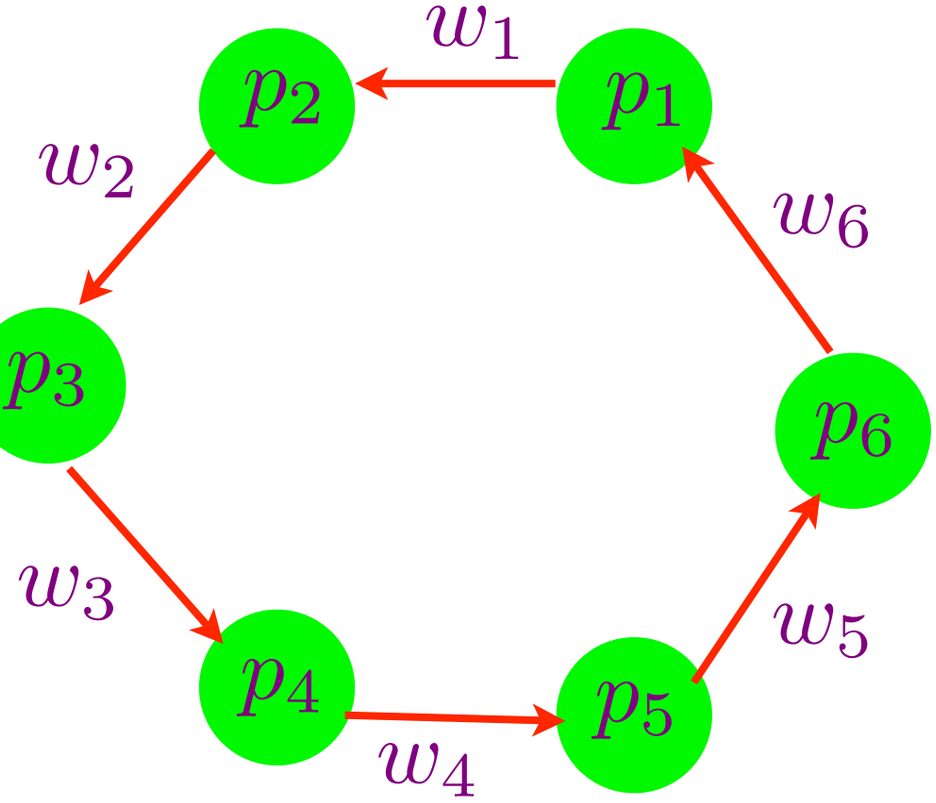
Stability Analysis: Weighted graph



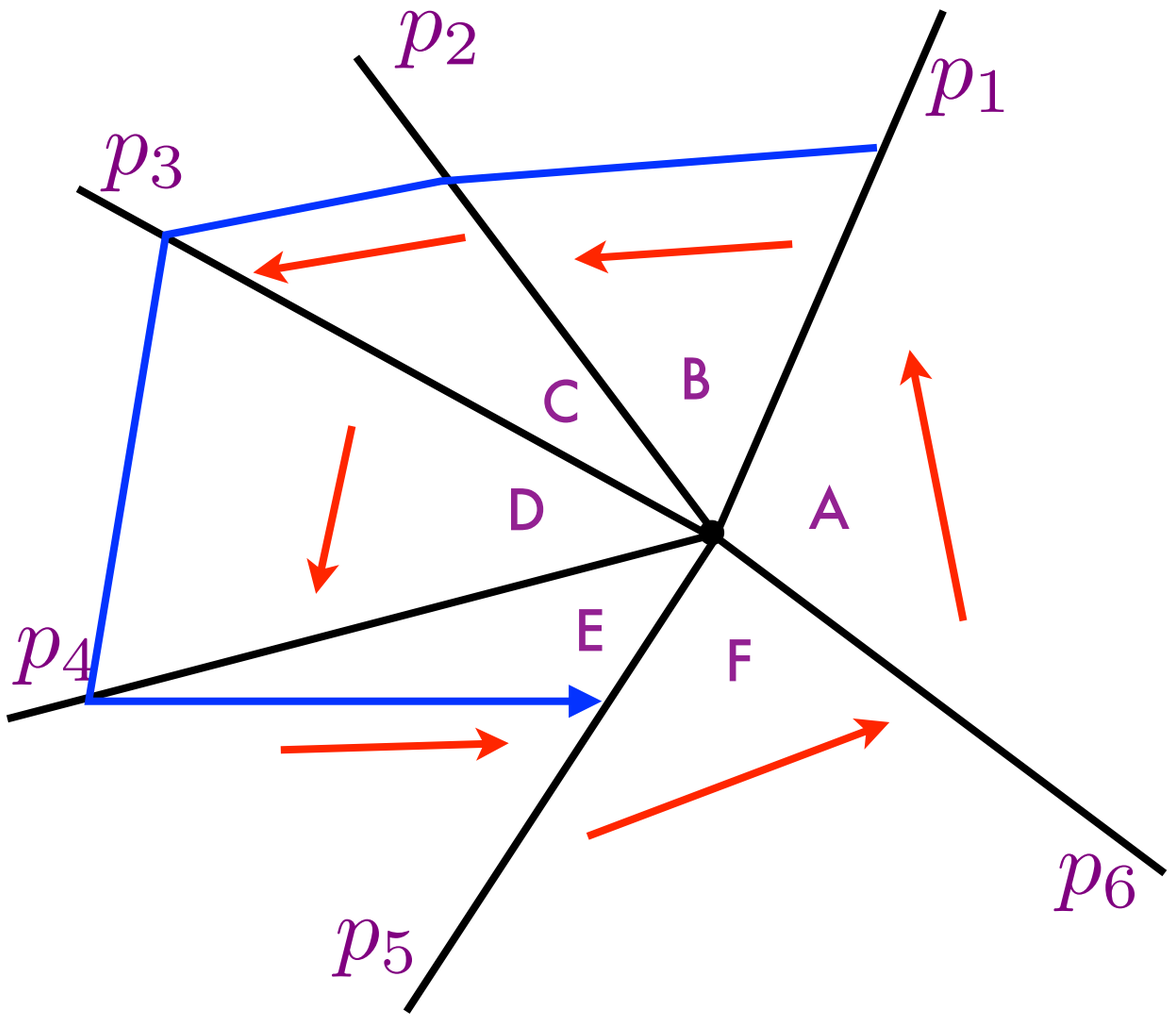


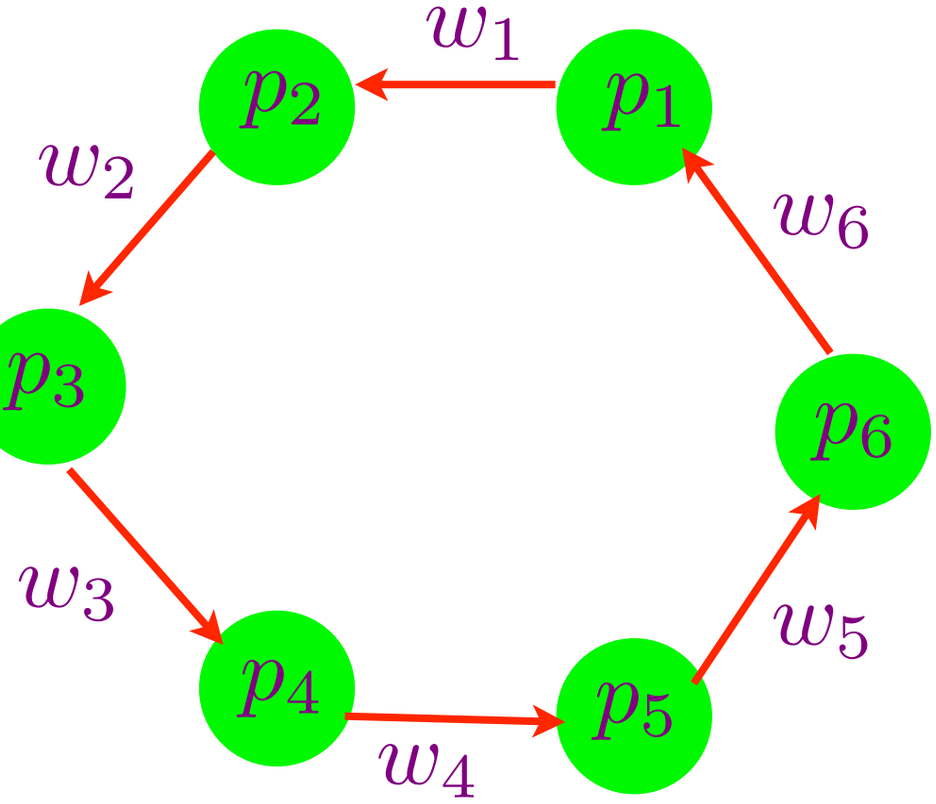
Stability Analysis: Weighted graph





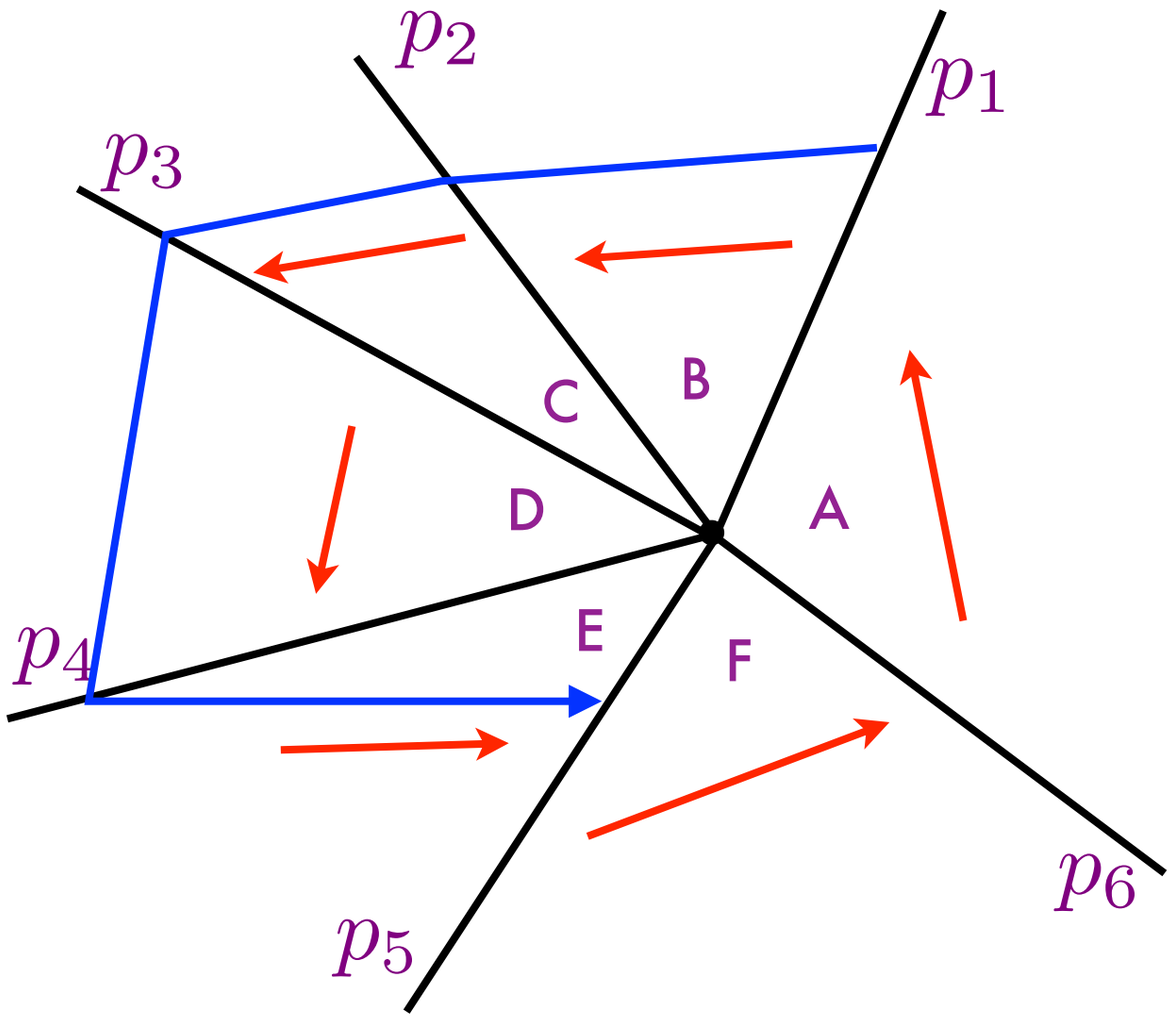
Stability Analysis: Weighted graph

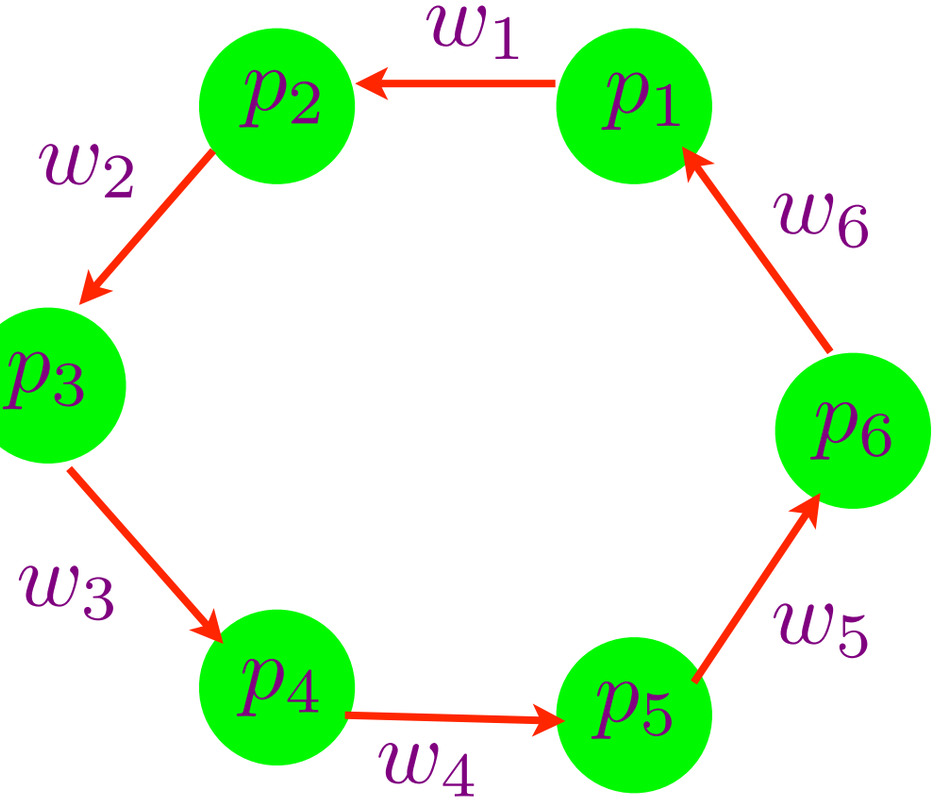




$$\sigma = \sigma(p_{i_1} p_{i_2}) \sigma(p_{i_2} p_{i_3}) \cdots \sigma(p_{i_{n-1}} p_{i_n})$$

Stability Analysis: Weighted graph

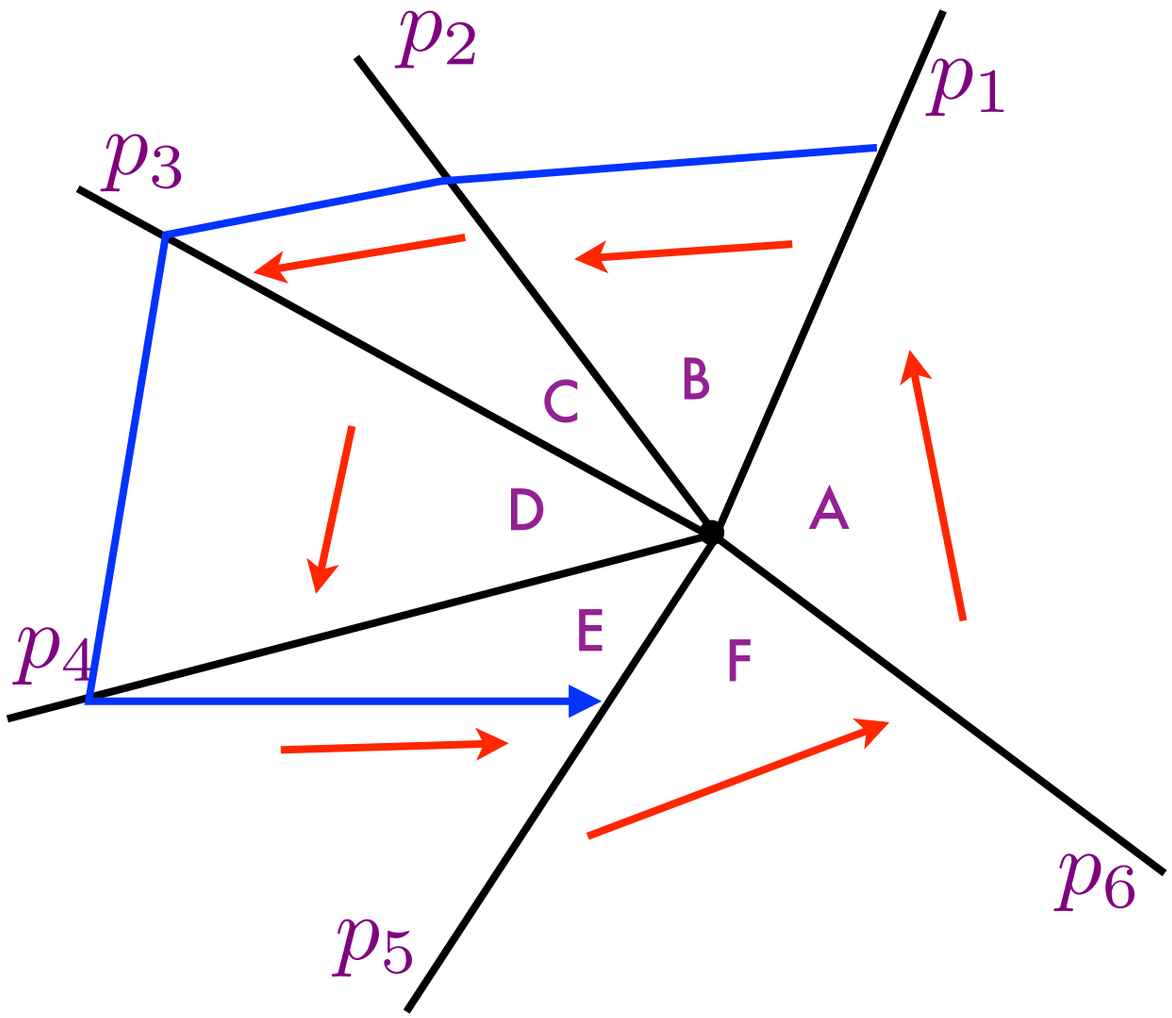


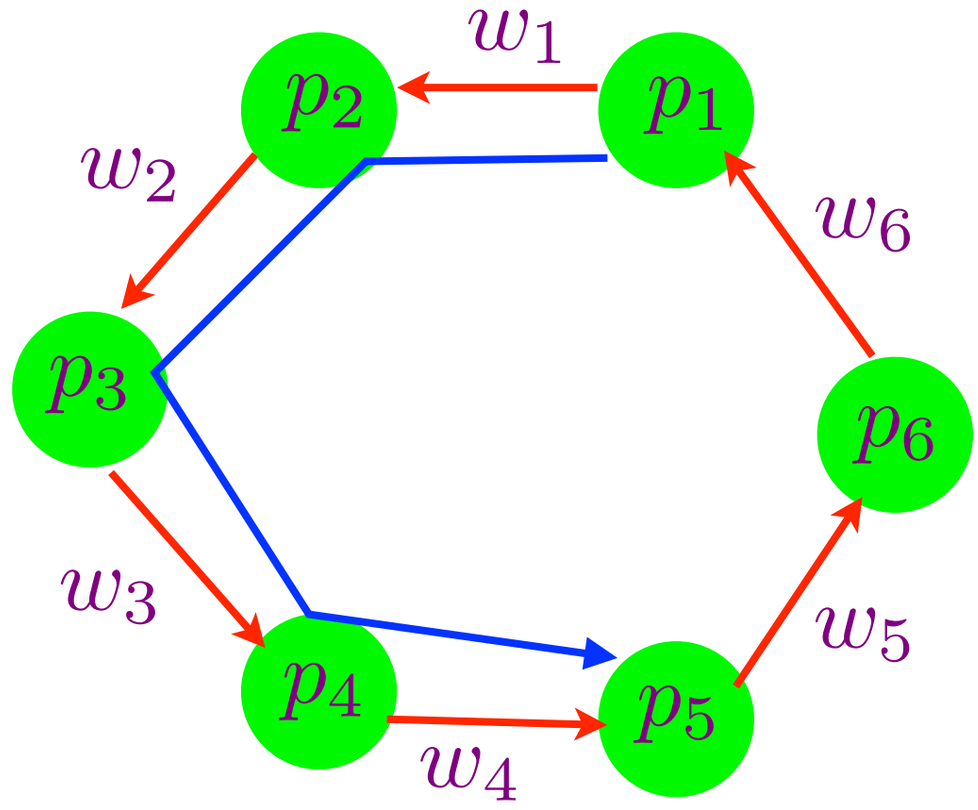


$$\sigma = \sigma(p_{i_1} p_{i_2}) \sigma(p_{i_2} p_{i_3}) \cdots \sigma(p_{i_{n-1}} p_{i_n})$$

$$w(\sigma) = \frac{d(\sigma(T))}{d(\sigma(0))}$$

Stability Analysis: Weighted graph

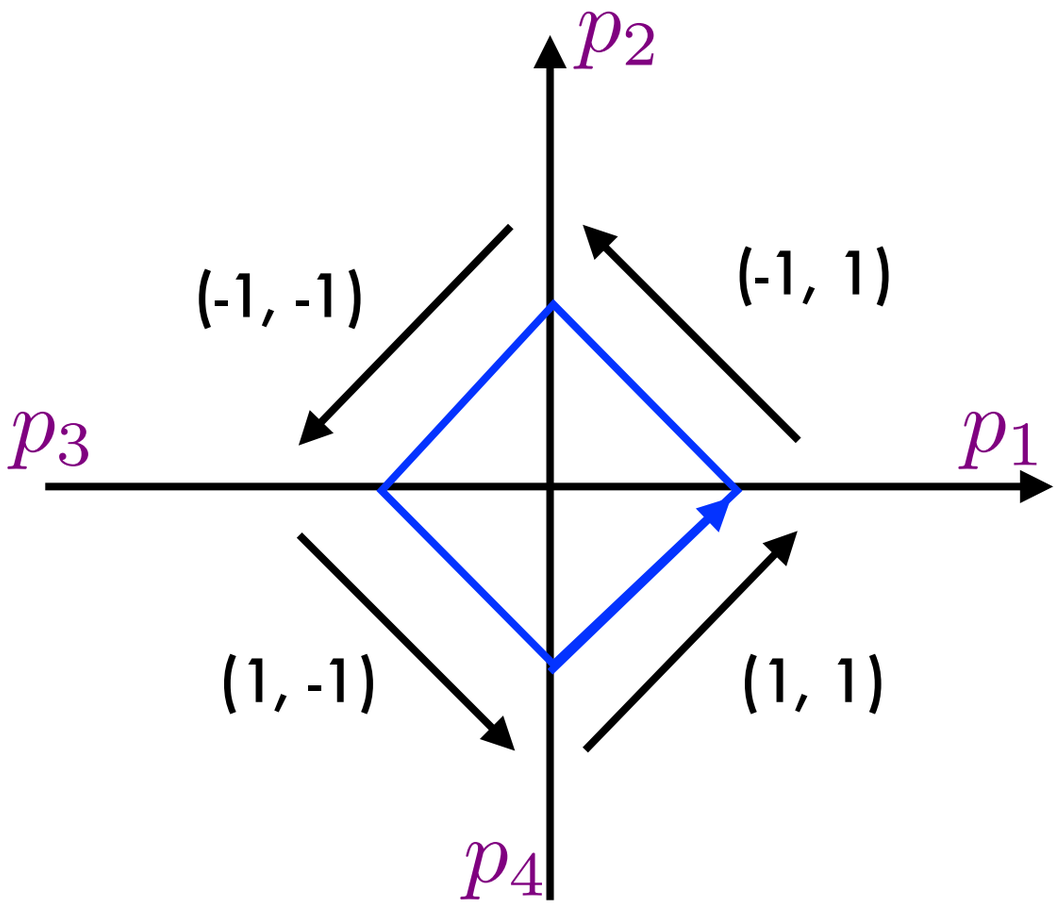




$$\sigma = \sigma(p_{i_1} p_{i_2}) \sigma(p_{i_2} p_{i_3}) \cdots \sigma(p_{i_{n-1}} p_{i_n})$$

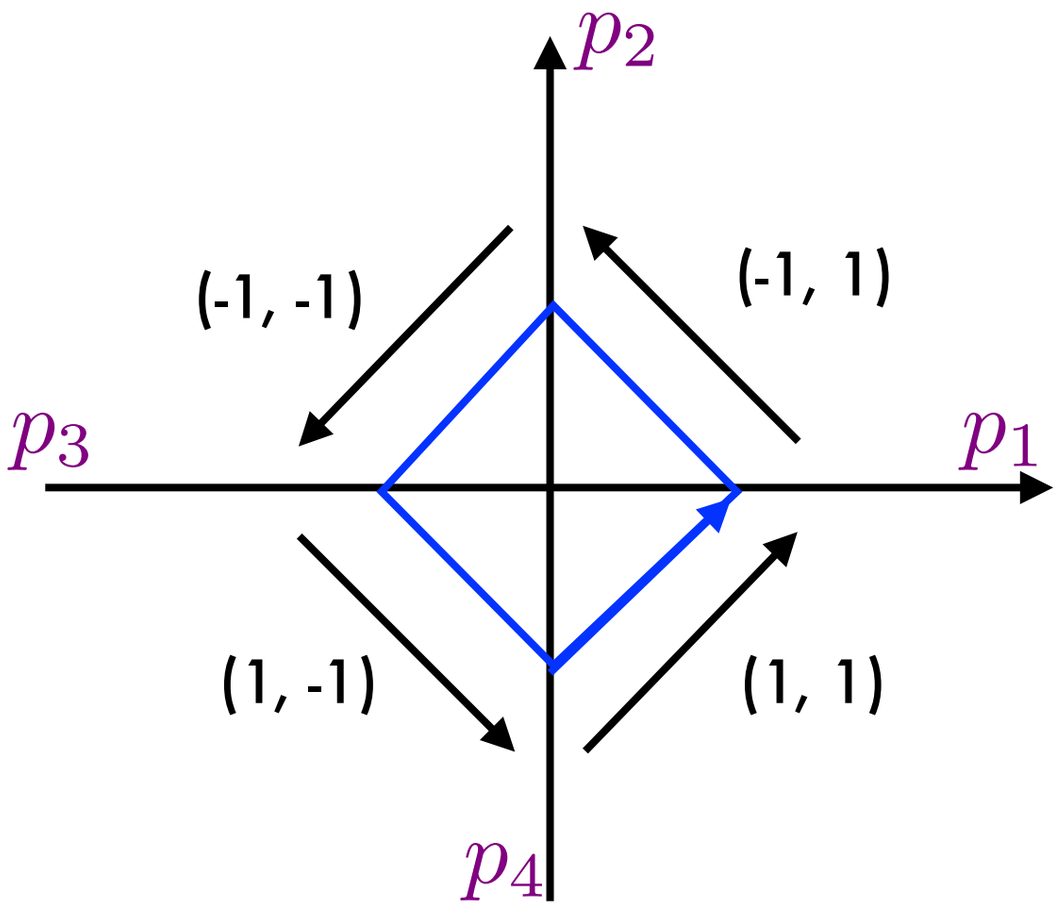
$$w(\sigma) = \frac{d(\sigma(T))}{d(\sigma(0))} = w(e_{i_1,i_2}) w(e_{i_2,i_3}) \cdots w(e_{i_{n-1},i_n})$$

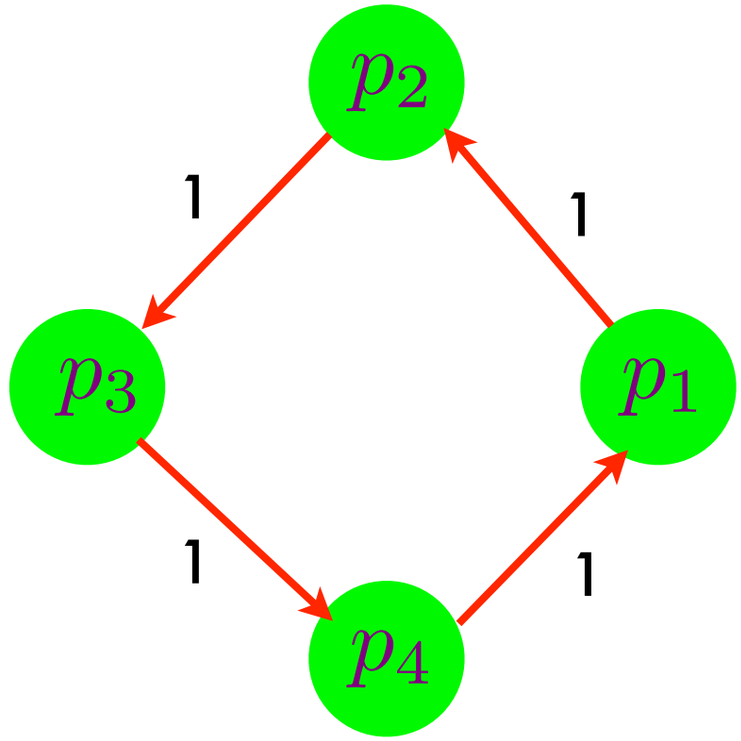
Stability Analysis: Example 1



Lyapunov stable, but not asymptotically stable

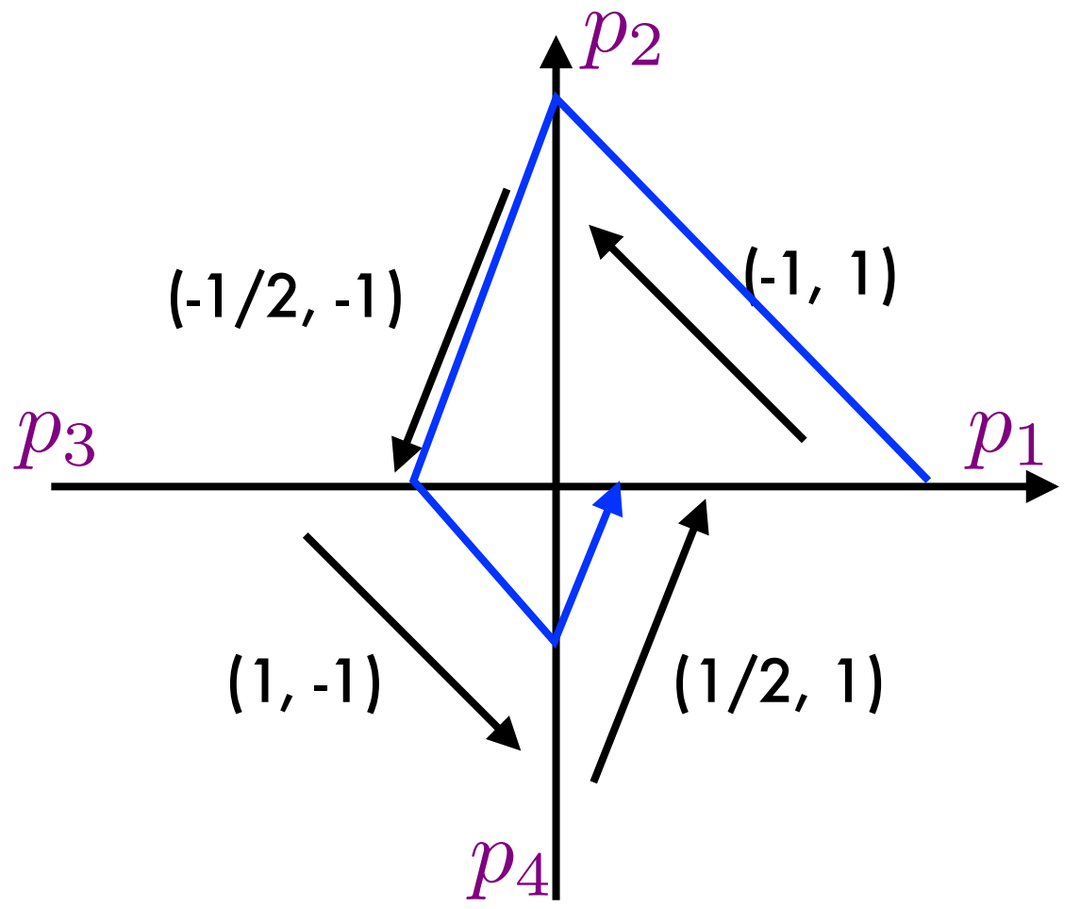
Stability Analysis: Example 1





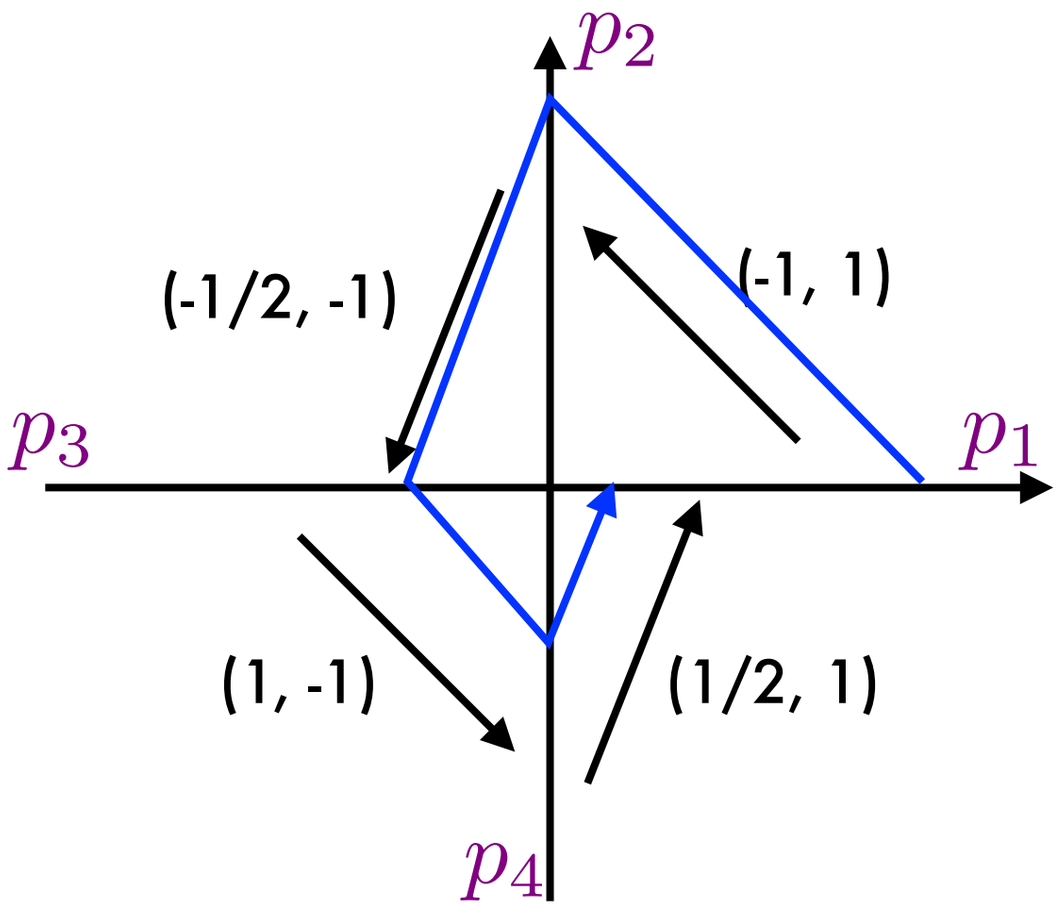
Lyapunov stable, but not asymptotically stable

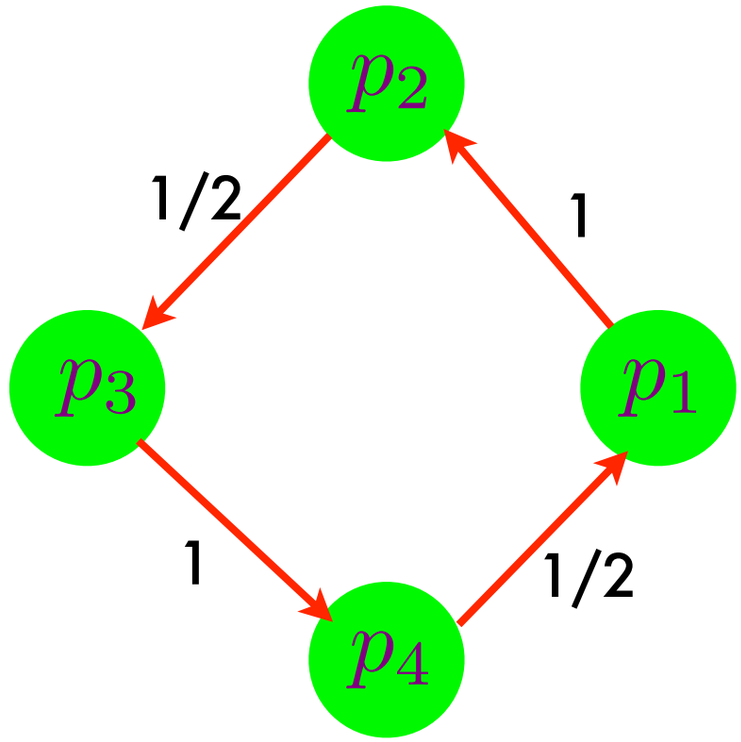
Stability Analysis: Example 2



Lyapunov stable and asymptotically stable

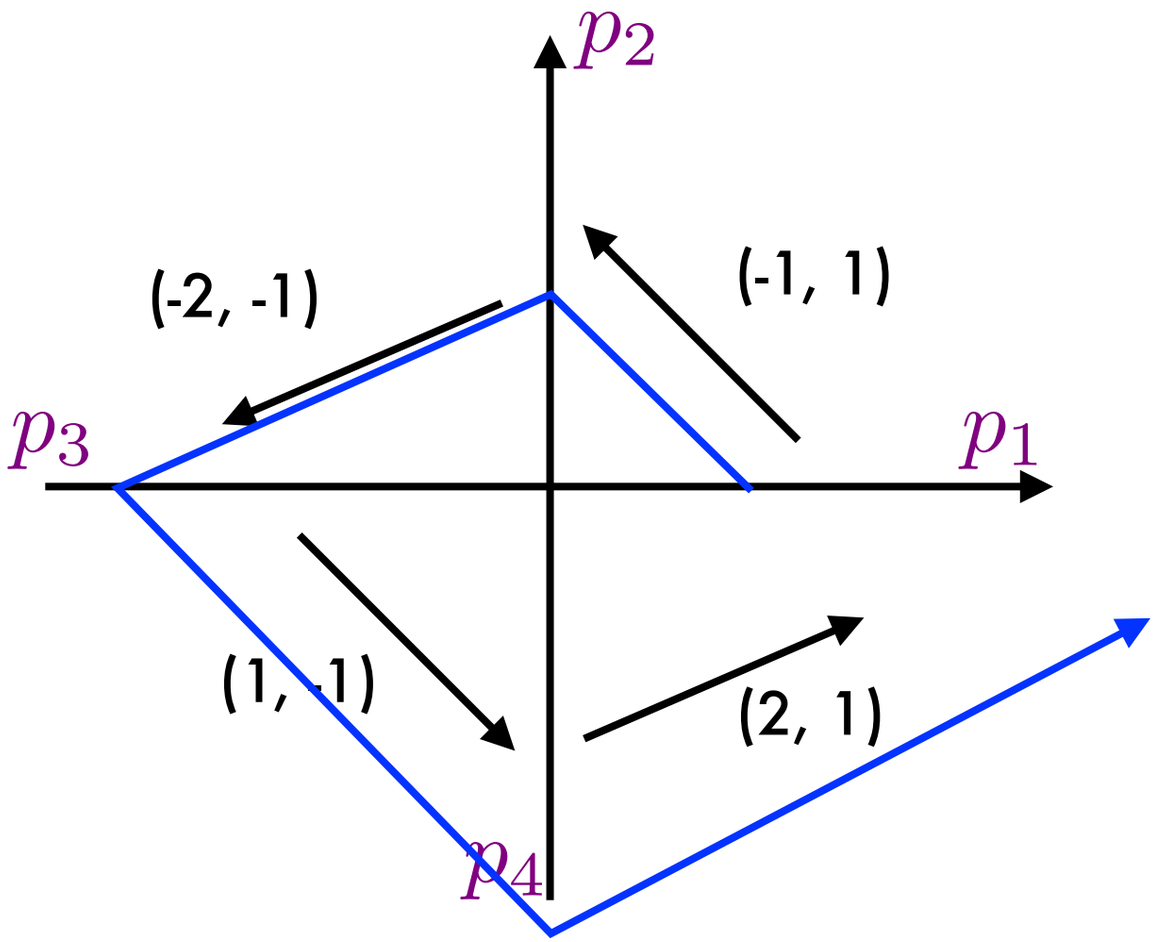
Stability Analysis: Example 2





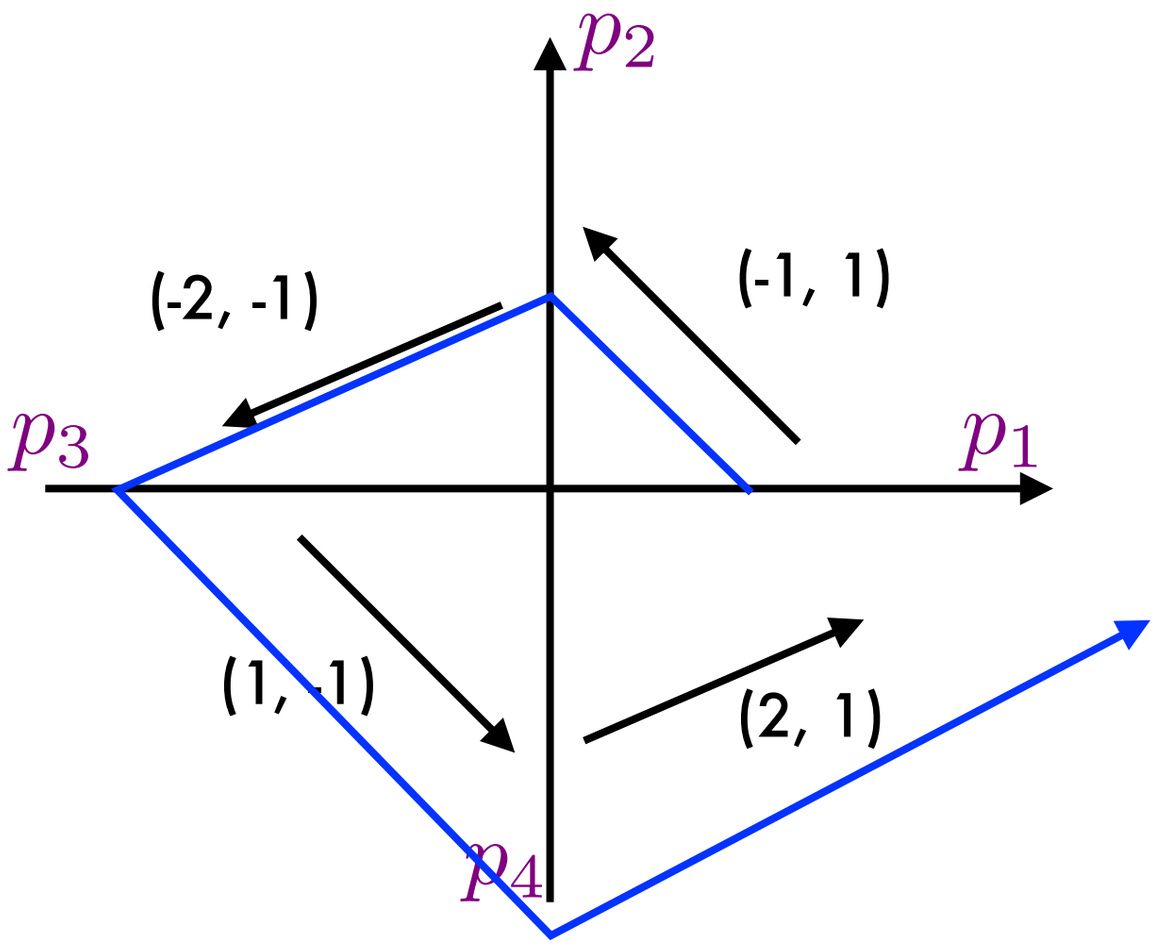
Lyapunov stable and asymptotically stable

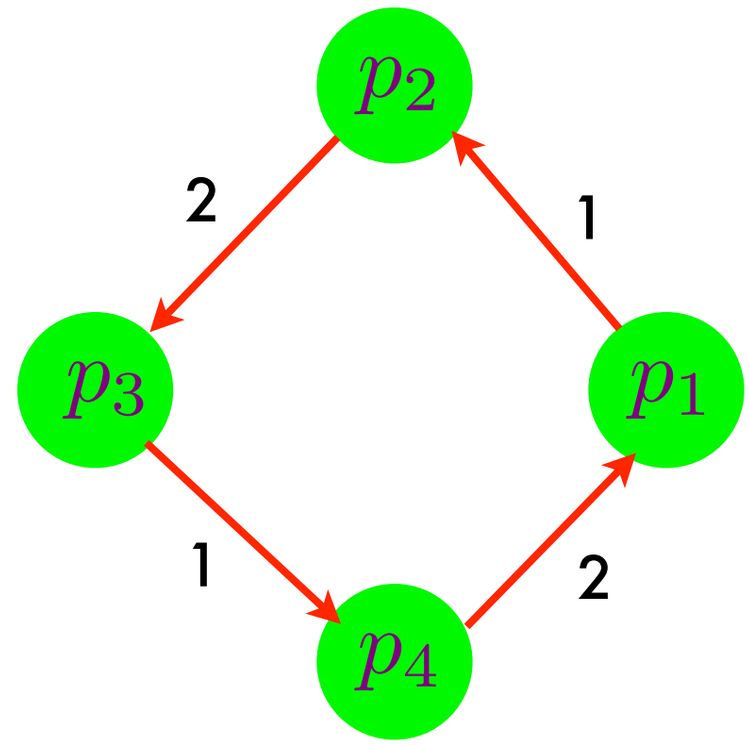
Stability Analysis: Example 3



Neither Lyapunov stable nor asymptotically stable

Stability Analysis: Example 3





Neither Lyapunov stable nor asymptotically stable

Stability Analysis: PCD

Theorem: (Lyapunov stability)

A Piecewise Constant Derivative System is Lyapunov stable

if

the weighted graph does not contain any cycles with the product of weights > 1

Stability Analysis: PCD

Theorem: (Lyapunov stability)

A Piecewise Constant Derivative System is Lyapunov stable

if

the weighted graph does not contain any cycles with the product of weights > 1

Theorem: (Asymptotic stability)

A Piecewise Constant Derivative System is asymptotically stable

if

the weighted graph does not contain any cycles with the product of weights >= 1

Stability Analysis Tools

Stability Analysis Tools

- SOSTOOLS - Sum-of-squares programming

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- LMI solvers - CVX

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- SOSTOOLS - Sum-of-squares programming
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- Stability Analysis: Stabhyli

Summary

- Hybrid Systems verification is challenging
  - Undecidable for simple subclasses
  - Standard techniques from the purely discrete and continuous worlds do not extend in a straightforward manner
- Model-checking & Deductive verification
- Approximation techniques
  - Safety - Predicate abstraction & CEGAR, hybridization
  - Stability - Linearization, Predicate abstraction

Some research directions

- Scalability
  - Efficient data structures
  - Approximation methods
  - Compositional analysis
- Applications
  - Exploiting structures
- Bridging the gap between model and implementation